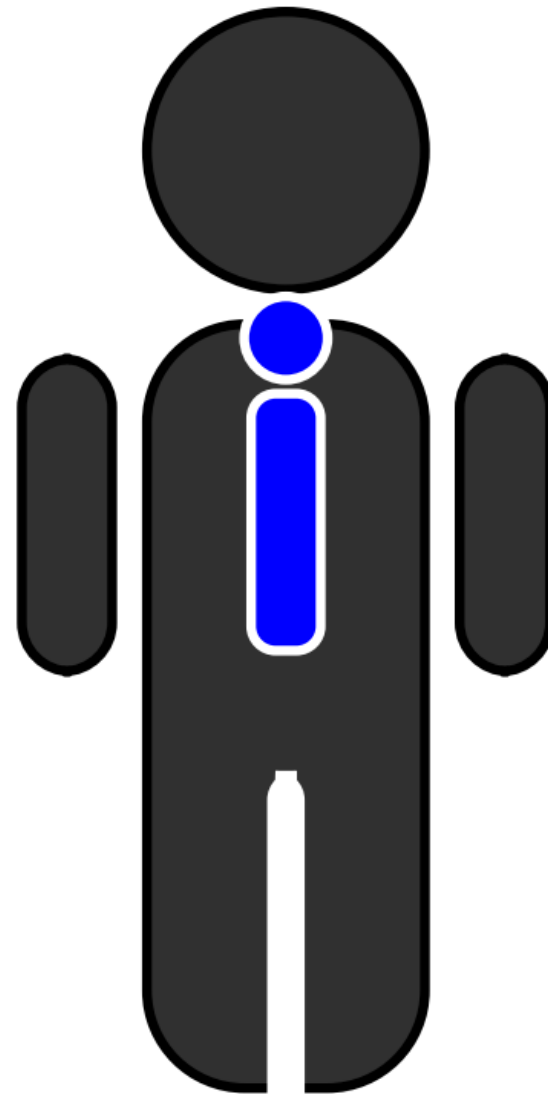


# 3. Numbers & Operations



# Chapter 3 - Numbers & Operations

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# INTRODUCTION

When learning and teaching about arithmetic, it helps to have mental and physical *models* for what the operations mean. That way, when you are presented with an unfamiliar problem or a question about why something is true, you can often work it out using the model — this might mean drawing pictures, using physical materials (manipulatives), or just thinking about the model to help you reason out the answer.

**Think/Pair/Share.** Write down your mental models for each of the four basic operations. What do they actually *mean*? How would you explain them to a second grader? What pictures could you draw for each operation? Think about each one separately, as well as how they relate to each other:

- addition
- subtraction
- multiplication, and
- division.

After writing down your own ideas, share them with a partner. Do you and your partner have the same models for each of the operations or do you think about them differently?

Teachers should have lots of mental models — lots of ways to explain the same concept. In this chapter, we'll look at some different ways to understand the four basic arithmetic operations. First, let's define some terms:

**Definition 1.1** *Counting numbers* are literally the numbers we use for counting: 1, 2, 3, 4, 5, .... These are sometimes called the *natural numbers* by mathematicians, and they are represented by the symbol  $\mathbb{N}$ .

*Whole numbers* are the counting numbers together with zero.

*Integers* include the positive and negative whole numbers, and mathematicians represent these with the symbol  $\mathbb{Z}$ . (This comes from German, where the word for “number” is “zählen.”)

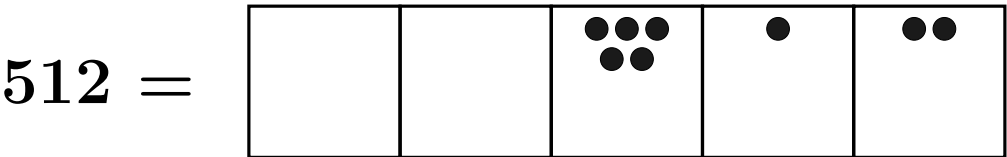
# DOTS AND BOXES

We already have a natural model for thinking about counting numbers: a number is a quantity of dots. Depending on which number system you use, you might write down the number in different ways. But the quantity of dots is a counting number, however you write it down.

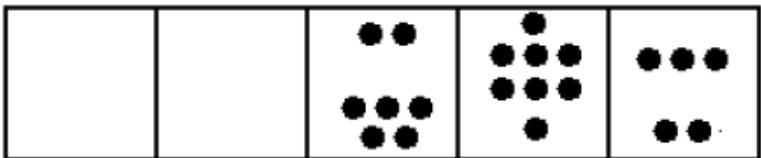
**2.1 Addition as combining.** For now, we'll focus on the base-10 system. Here's how we think about the number 273 in that system:



And here is the number 512:



*Example 2.1* (273+512). We can add these in the natural way: just combine the piles of dots. Since they're already in place-value columns, we can combine dots from the two numbers that are in the same place-value box.



We can count up the answer: there are 7 dots in the hundreds box, 8 dots in the tens box, and 5 dots in the ones box.

$$\begin{array}{r} 273 \\ + 512 \\ \hline 785 \end{array}$$

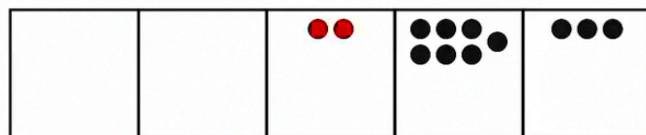
And saying out the long way we have:

- Two hundreds plus five hundreds gives 7 hundreds.
- Seven tens plus one ten gives 8 tens.
- Three ones plus two ones gives 5 ones.

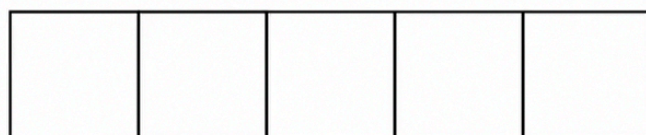
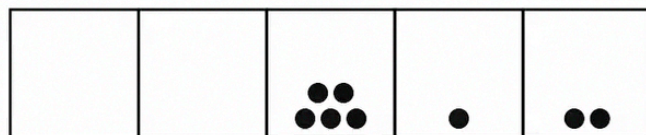
This gives the answer: 785.

**Movie 1** Example 2.1 (273+512)

273 =



+ 512 =



$$\begin{array}{r} 163 \\ + 489 \\ \hline 5 \quad 14 \quad 12 \end{array}$$

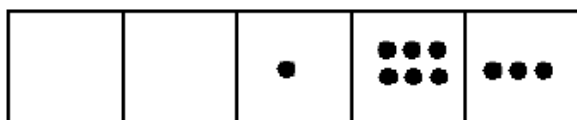
And this is absolutely mathematically correct:

- One hundred plus four hundreds does give 5 hundreds.
- Six tens plus eight tens does give 14 tens.
- Three ones plus nine ones does give 12 ones.

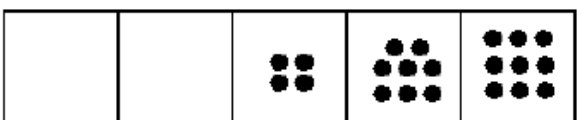
The answer is 5 | 14 | 12, which we might try to pronounce as “five hundred and fourteeny-teny twelvety!” (Oh my!)

Example 2.2 (163+489). Let’s do another one. Consider 163+489.

163 =



+ 489 =



$$= 5 \mid 14 \mid 12$$

The trouble with this answer is that most of the rest of the world wouldn’t understand what we are talking about! Since this is a base 10 system, we can do some explosions.

$$= 6 \mid 4 \mid 2$$

The answer now looks like “six hundred forty twelvety.” That’s still not a familiar number, so let’s do another explosion.

$$= 6 \mid 5 \mid 2$$

The answer is “six hundred fifty two.” Okay, the world can understand this one!

$$\begin{array}{r} 163 \\ + 489 \\ \hline \end{array}$$

5 | 14 | 12 = 652

**Movie 2** Example 2.2 (163+489)

$$\begin{array}{r} 163 = \\ + 489 = \\ \hline = \end{array}$$

**Think/Pair/Share.** Solve the following exercises by thinking about the dots and boxes. (You can draw the pictures or just imagine them.) Then translate the answer into something the rest of the world can understand.

$$\begin{array}{r} 148 \\ + 323 \\ \hline \end{array} \quad \begin{array}{r} 567 \\ + 271 \\ \hline \end{array} \quad \begin{array}{r} 377 \\ + 188 \\ \hline \end{array} \quad \begin{array}{r} 582 \\ + 714 \\ \hline \end{array}$$

$$\begin{array}{r} 310462872 \\ + 389107123 \\ \hline \end{array} \quad \begin{array}{r} 872637163 \\ + 187782748 \\ \hline \end{array}$$

**Problem 1.** Use the dots and boxes technique to solve these problems. *Do not convert to base 10! Try to work directly in the base given.* It might help to actually draw the pictures.

$$\begin{array}{r} 20413_{\text{five}} \\ + 13244_{\text{five}} \\ \hline \end{array}$$

$$\begin{array}{r} 4052_{\text{nine}} \\ + 6288_{\text{nine}} \\ \hline \end{array}$$

$$\begin{array}{r} 3323_{\text{seven}} \\ + 3555_{\text{seven}} \\ \hline \end{array}$$

**2.2 The Standard Algorithm for Addition.** Let's go back to the example 163+489. Some teachers don't like writing:

$$\begin{array}{r} 163 \\ + 489 \\ \hline \end{array}$$

5 | 14 | 12 = 652

They prefer to teach their students to start with the 3 and 9 at the end and sum those to get 12. This is of course correct — we got 12 as well.

$$\begin{array}{r} 163 = \\ + 489 = \\ \hline \end{array}$$

↓

$$\begin{array}{r} \phantom{163} \\ \phantom{+ 489} \\ \phantom{\hline} \end{array} = \phantom{163} \phantom{+ 489} \phantom{\hline} 12$$

But they don't want students to write or think "twelvety," so they have their students write something like this:

$$\begin{array}{r} 1 \\ 163 \\ + 489 \\ \hline 2 \end{array}$$

which can seem completely mysterious. What's really going on? They are exploding ten dots, of course!

$$\begin{array}{l} = \begin{array}{|c|c|c|c|c|} \hline & & & & \bullet\bullet\bullet\bullet \\ \hline \end{array} = \begin{array}{|c|} \hline 12 \\ \hline \end{array} \\ \\ = \begin{array}{|c|c|c|c|c|} \hline & & & \bullet & \bullet\bullet \\ \hline \end{array} = \begin{array}{|c|c|} \hline 1 & 2 \\ \hline \end{array} \end{array}$$

Now we carry on with the problem and add the tens. Students are taught to write:

$$\begin{array}{r} 1 \\ 163 \\ + 489 \\ \hline 52 \end{array}$$

But what this means is better shown in this next picture:

$$\begin{array}{r} 163 = \begin{array}{|c|c|c|c|c|} \hline & & \bullet & \bullet\bullet\bullet & \bullet\bullet\bullet \\ \hline \end{array} \\ + 489 = \begin{array}{|c|c|c|c|c|} \hline & & \bullet\bullet & \bullet\bullet\bullet & \bullet\bullet\bullet \\ \hline \end{array} \\ \hline \begin{array}{|c|c|c|c|c|} \hline & & & \bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet & \bullet\bullet \\ \hline \end{array} = \begin{array}{|c|} \hline 15 \\ \hline \end{array} 2 \\ \text{14 dots} \quad \text{The 1 dot from before} \\ \\ = \begin{array}{|c|c|c|c|c|} \hline & & \bullet & \bullet\bullet\bullet & \bullet\bullet \\ \hline \end{array} = \begin{array}{|c|c|} \hline 1 & 5 \\ \hline \end{array} 2 \end{array}$$

And now we finish the problem:

$$\begin{array}{r} 163 = \begin{array}{|c|c|c|c|c|} \hline & & \bullet & \bullet\bullet\bullet & \bullet\bullet\bullet \\ \hline \end{array} \\ + 489 = \begin{array}{|c|c|c|c|c|} \hline & & \bullet\bullet & \bullet\bullet\bullet & \bullet\bullet\bullet \\ \hline \end{array} \\ \hline \begin{array}{|c|c|c|c|c|} \hline & & \bullet\bullet\bullet & \bullet\bullet\bullet & \bullet\bullet \\ \hline \end{array} = \begin{array}{|c|c|} \hline 6 & 5 \\ \hline \end{array} 2 \end{array}$$

$$\begin{array}{r} 1 \\ 163 \\ + 489 \\ \hline 652 \end{array}$$



In the standard algorithm, we work from right to left, doing the “explosions” as we go along. This means that we start adding at the ones place and work towards the left-most place value, “carrying” digits that come from the explosions. (This is really not carrying; a better term for it is *regrouping*. Ten ones become one ten. Ten tens become one hundred. And so on.)

In the dots and boxes method, we add in any direction or order we like and then we do the explosions at the end.

- **Why do we like the standard algorithm?** Because it is efficient.
- **Why do we like the dots and boxes method?** Because it is easy to understand.

**Think/Pair/Share.** Redo the problems below using the standard algorithm. You will see that it is quicker.

$$\begin{array}{r} 148 \\ + 323 \\ \hline \end{array}$$

$$\begin{array}{r} 567 \\ + 271 \\ \hline \end{array}$$

$$\begin{array}{r} 377 \\ + 188 \\ \hline \end{array}$$

$$\begin{array}{r} 582 \\ + 714 \\ \hline \end{array}$$

$$\begin{array}{r} 310462872 \\ + 389107123 \\ \hline \end{array}$$

$$\begin{array}{r} 872637163 \\ + 187782748 \\ \hline \end{array}$$

**2.3. Subtraction as take-away.** In addition, we started with two collections of dots (two numbers), and we *combined* them to form one bigger collection. That’s pretty much the definition of addition: combining two collections of objects. In subtraction, we start with one collection of dots (one number), and we take some dots away.

*Example 2.3 (376-125).* Suppose we want to find 376-125 in the dots and boxes model. We start with the representation of 376:

$$376 = \begin{array}{|c|c|c|c|c|} \hline & & \bullet\bullet\bullet & \bullet\bullet\bullet\bullet\bullet & \bullet\bullet\bullet \\ \hline \end{array}$$

Since we want to “take away” 125, that means we take away one dot from the hundreds box, leaving two dots. We take away two dots from the tens box, leaving five dots. And we take away five dots from the ones box, leaving one dot.

**Movie 3** *Example 2.3 (376-125)*

$$376 - 125 = \begin{array}{|c|c|c|c|c|} \hline & & & & \\ \hline \end{array}$$



So the answer is:

$$\begin{array}{r} 376 \\ - 125 \\ \hline 251 \end{array}$$

And saying it out the long way we have:

- Three hundreds take away one hundred leaves 2 hundreds.
- Seven tens take away two tens gives 5 tens.
- Six ones take away five ones gives 1 one.

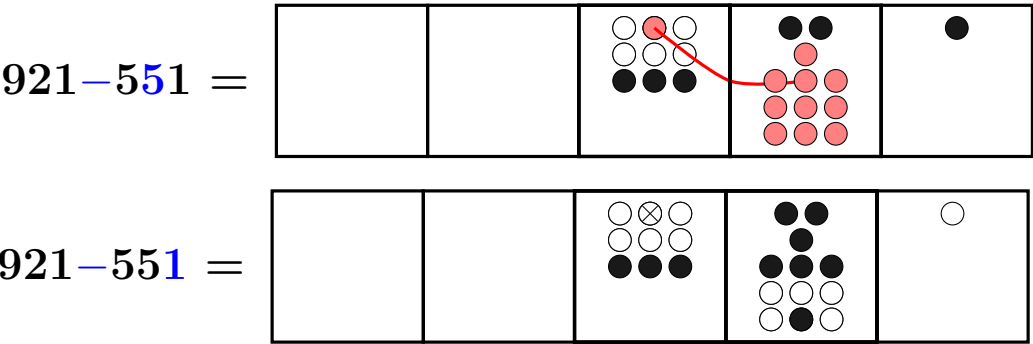
*Example 2.4* (921-551). Let’s try a somewhat harder example: 921-551. We start with the representation of 921:



Since we want to “take away” 551, that means we take away five dots from the hundreds box, leaving four dots.



Now we want to take away five dots from the tens box, but we can’t do it! There are only two dots there. What can we do? Well, we still have some hundreds, so we can “unexplode” a hundreds dot, and put ten dots in the tens box instead. Then we’ll be able to take five of them away, leaving seven.



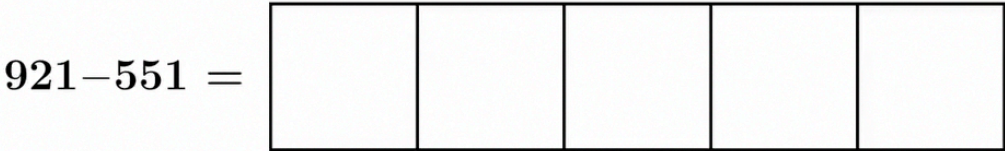
(Notice that we also have one less dot in the hundreds box; there’s only three dots there now.)

Now we want to take one dot from the ones box, and that leaves no dots there.

So the answer is:

$$\begin{array}{r} 921 \\ - 551 \\ \hline 370 \end{array}$$

**Movie 4** *Example 2.4* (921-551)



**Think/Pair/Share.** Solve the following exercises by thinking about the dots and boxes. (You can draw pictures, or just imagine them.)

323

– 148

567

– 271

377

– 188

714

– 582

389107123

– 310462872

872637163

– 187782748

**Problem 2.** Use the dots and boxes technique to solve these problems. *Do not covert to base 10! Try to work directly in the base given.* It might help to actually draw the pictures.

20413<sub>five</sub>

– 13244<sub>five</sub>

6288<sub>nine</sub>

– 4052<sub>nine</sub>

3555<sub>seven</sub>

– 3323<sub>seven</sub>

**2.4 The Standard Algorithm for Subtraction.** Just like in addition, the standard algorithm for subtraction requires you to work from right to left, and “borrow” (this is really *re-grouping*!) whenever necessary. Notice that in the dots and boxes approach, you don’t need to go in any particular order when you do the subtraction. You just “unexplode” the dots as necessary when computing.

Here’s how the standard algorithm looks with the dots and boxes model for 921 - 551: Start with 921 dots.

921 =

•••••

•••••

••

•

Then take away one dot from the ones box.

921–551 =

•••••

•••••

••

○

921

– 551

0

Now we want to take away five dots from the tens box. But there aren’t five dots there. So we “unexplode” one of the hundreds dots to get more tens:

921–551 =

•••••

•••••

•••••

•••••

○

921–551 =

⊗

•••••

•••••

○•○•○

○

$$\begin{array}{r} 8\ 12 \\ \cancel{9}\ \cancel{2}\ 1 \\ -\ 5\ 5\ 1 \\ \hline 7\ 0 \end{array}$$

Finally, we want to take away five from the eight dots left in the hundreds column.

$$\begin{array}{r} 8\ 12 \\ \cancel{9}\ \cancel{2}\ 1 \\ -\ 5\ 5\ 1 \\ \hline 3\ 7\ 0 \end{array}$$

**Movie 5** *Example 2.4* (921-551, standard algorithm)

921-551 =

--	--	--	--	--

## 2.5. Multiplication as Repeated Addition

**Problem 3.** Jenny was asked to compute  $243192 \times 4$ . She wrote

$$243192 \times 4 = 8|16|12|4|36|8$$

- (a) What was Jenny thinking about? Is her answer correct?
- (b) Translate Jenny's answer into a number that the rest of the world can understand.
- (c) Use Jenny's method to find the answers to these multiplication exercises. Be sure to translate your answers into familiar base 10 numbers.

$$156 \times 3 = \quad 2873 \times 2 = \quad 71181 \times 5 = \quad 3726510392 \times 2 =$$

**Problem 4.** Can you adapt Jenny's method to solve these problems? Write your answers in base eight. Try to work directly in base eight rather than converting to base 10 and back again!

$$156_{\text{eight}} \times 3_{\text{eight}} =$$

$$2673_{\text{eight}} \times 4_{\text{eight}} =$$

$$36255772_{\text{eight}} \times 2_{\text{eight}} =$$

**Think/Pair/Share.** After you have worked on problems 3 and 4, share your ideas with a partner. Can you relate Jenny's method to the standard algorithm for multiplication?

Jenny might have been thinking about multiplication as repeated addition. If we have some number  $N$  and we multiply that number by 4, what we mean is:

$$4 \cdot N = N + N + N + N.$$

If we take the number 243192 and add it to itself four times using the “combining method,” we get  $2 + 2 + 2 + 2 = 8$  ones,  $9 + 9 + 9 + 9 = 36$  tens,  $1 + 1 + 1 + 1 = 4$  hundreds, and so on.

**Notation:** Notice that we have used both  $\times$  and  $\cdot$  to represent multiplication. It’s a bit awkward to use  $\times$  when you’re also using variables. Is it the letter x? Or the multiplication symbol  $\times$ ? It can be hard to tell! In this case, the symbol  $\cdot$  is more clear. We can even simplify the notation further, writing  $4N$  instead of  $4 \cdot N$ . But of course we only do that when we are multiplying *variables* by some quantity. (We wouldn’t want  $34$  to mean  $3 \cdot 4$ , would we?)

**Problem 5.** Here is a strange addition table. Use it to solve the following problems. Important: Don’t try to assign numbers to A, B, and C. Solve the problems just using what you know about the operations!

+	A	B	C
A	C	A	B
B	A	B	C
C	B	C	A

$A + B$        $B + C$        $2A$        $5C$        $3A + 4B$

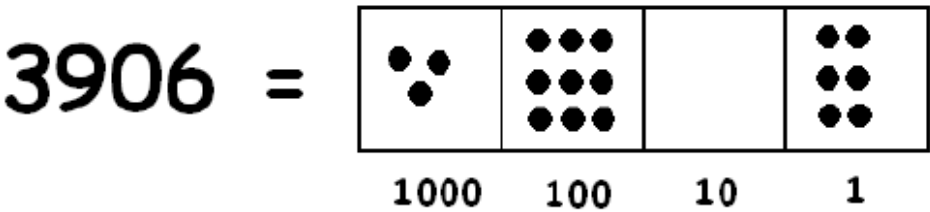
**Think/Pair/Share.** Discuss your answers with a partner. How does an addition table help you solve multiplication problems like  $5C$ ?

**2.6. Quotative Model of Division.** Suppose you are asked to compute  $3906 \div 3$ . One way to interpret this question (there are others) is:

“How many groups of 3 fit into 3906?”

**Definition 2.5.** In the quotative model of division, you are given a dividend (here it is 3906), and you are asked to split it into equal-sized groups, where the size of the group is given by the divisor (here it is 3).

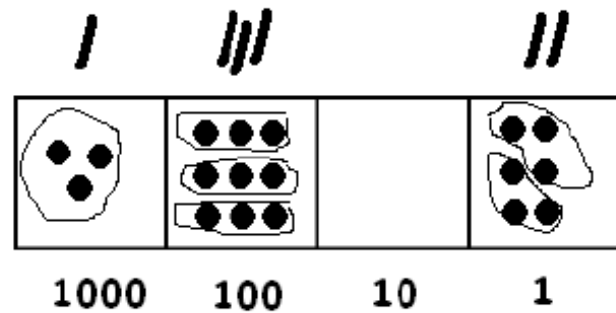
In our dots and boxes model, the dividend 3906 looks like this:



and three dots looks like this: •••. So we are really asking:

“How many groups of ••• fit into the picture of 3906?”

There is one group of 3 at the thousands level, and three at the hundreds level, none at the tens level, and two at the ones level.



Notice what we have in the picture:

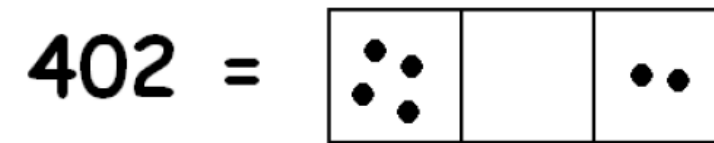
- One group of 3 in the thousands box.
- Three groups of 3 in the hundreds box.
- Zero groups of 3 in the tens box.
- Two groups of 3 in the ones box.

This shows that 3 goes into 3906 one thousand, three hundreds and two ones times. That is,

$$3906 \div 3 = 1302.$$

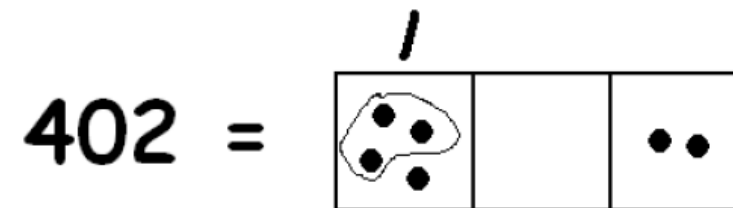
**Fun fact:** The division sign  $\div$  has an unusual name. It is called an *obelus*. Not many people know this.

Let's try a harder one! Consider  $402 \div 3$ . Here's the picture:



We are still looking for groups of three dots: ●●●

There is certainly one group at the 100s level.



and now it seems we are stuck there are no more groups of three!

**Think/Pair/Share.** What can we do now? Are we really stuck? Can you finish the division problem?

*Example 2.6* ( $402 \div 3$ ). Here are the details worked out for  $402 \div 3$ . But don't read this until you've thought about it yourself!

Since each dot is worth ten dots in the box to the right we can write:

$$402 = \begin{array}{|c|c|c|} \hline / & & \\ \hline \text{4 dots} & \text{0 dots} & \text{2 dots} \\ \hline \end{array}$$

Now we can find more groups of three:

$$402 = \begin{array}{|c|c|c|} \hline / & // & \\ \hline \text{4 dots} & \text{0 dots} & \text{2 dots} \\ \hline \end{array}$$

There is still a troublesome extra dot. Let's unexplode it too

$$402 = \begin{array}{|c|c|c|} \hline / & // & \\ \hline \text{4 dots} & \text{0 dots} & \text{2 dots} \\ \hline \end{array}$$

This gives us more groups of three:

$$402 = \begin{array}{|c|c|c|} \hline / & // & /// \\ \hline \text{4 dots} & \text{0 dots} & \text{2 dots} \\ \hline \end{array}$$

In the picture we have:

- One group of 3 in the hundreds box.
- Three groups of 3 in the tens box.
- Four groups of 3 in the ones box.

Finally we have the answer!

$$402 \div 3 = 134.$$

**Think/Pair/Share.** Solve each of these exercises using the dots and boxes method:

$$62124 \div 3 \quad 61230 \div 5$$

Let's turn up the difficulty a notch. Consider  $156 \div 12$ . Here we are looking for groups of 12 in this picture:

$$156 = \begin{array}{|c|c|c|} \hline \text{1 dot} & \text{5 dots} & \text{6 dots} \\ \hline \end{array}$$



What does 12 look like? It can be twelve dots in a single box:

$$12 = \begin{array}{|c|} \hline \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \\ \hline \end{array}$$

But most often we would write 12 this way:

$$12 = \begin{array}{|c|c|} \hline \bullet & \bullet \bullet \\ \hline \end{array}$$

We certainly see some of these in the picture. There is certainly one at the tens level:

$$156 = \begin{array}{|c|c|c|} \hline \bullet & \bullet \bullet \bullet & \bullet \bullet \bullet \\ \hline \end{array}$$

(**Remember:** With an unexplosion this would be twelve dots in the tens box.) And three at the ones level:

$$156 = \begin{array}{|c|c|c|} \hline \bullet & \bullet \bullet \bullet & \bullet \bullet \bullet \\ \hline \end{array}$$

So in the picture we have:

- One group of 12 dots in the tens box.
- Three groups of 12 dots in the ones box.

That means

$$156 \div 12 = 13.$$

**Problem 6.** Use the dots and boxes model to compute each of the following:

$$13453 \div 11$$

$$4853 \div 23$$

$$214506 \div 102$$

**Think/Pair/Share.**

- Compare your solutions to problem 6 with a partner.
- When you agree that you are doing the process correctly, use dots and boxes to compute these:
 
$$2130 \div 10$$

$$41300 \div 100$$
- Discuss: What pictures did you use for 10 and for 100? Can you describe in words what happens when dividing by 10 and by 100 and why?

**2.7. The Standard Algorithm for Division.** We used dots and boxes to show that  $402 \div 3 = 134$ .

$$402 = \begin{array}{|c|c|c|} \hline \bullet & \bullet \bullet \bullet & \bullet \bullet \bullet \\ \hline \end{array}$$



In elementary school, you might have learned to solve this division problem by using a diagram like the following:

$$\begin{array}{r} 134 \\ 3 \overline{)402} \\ \underline{3\phantom{00}} \phantom{0} \\ 10\phantom{0} \\ \underline{9\phantom{0}} \phantom{0} \\ 12\phantom{0} \\ \underline{12} \\ 0 \end{array}$$

At first glance this seems very mysterious, but it is really no different from the dots and boxes method. Here is what the table means.

To compute  $402 \div 3$ , we first make a big estimation as to how many groups of 3 there are in 402. Let's guess that there are 100 groups of three.

	<b>Groups of 3</b>
$3 \overline{)402}$	<b>100</b>
<u>300</u>	

How much is left over after taking away 100 groups of 3? We subtract to find that there is 102 left.

	<b>Groups of 3</b>
$3 \overline{)402}$	<b>100</b>
<u>300</u>	
102	

How many groups of 3 are in 102? Let's try 30:

	<b>Groups of 3</b>
$3 \overline{)402}$	<b>100</b>
<u>300</u>	
<u>102</u>	
90	<b>30</b>

How many are left? There are 12 left and there are four groups of 3 in 12.

	<b>Groups of 3</b>
$3 \overline{)402}$	<b>100</b>
<u>300</u>	
<u>102</u>	
<u>90</u>	<b>30</b>
<u>12</u>	
<u>12</u>	<b>4</b>
0	

That accounts for entire number 402. And where do we find the final answer? Just add the total count of groups of three that we tallied:

$$402 \div 3 = 100 + 30 + 4 = 134.$$

**Think/Pair/Share.** Compare the two division diagrams below. In what way are they the same? In what way are they different?

Also look at the dots and boxes method. In what way is it the same or different from the two diagrams?

$$\begin{array}{r} 134 \\ 3 \overline{)402} \\ \underline{3 \phantom{00}} \\ 10 \phantom{0} \\ \underline{9 \phantom{0}} \\ 12 \\ \underline{12} \\ 0 \end{array}$$

$3 \overline{)402}$	Groups of 3
$\underline{300}$	100
$\underline{102}$	
$\underline{90}$	30
$\underline{12}$	
$\underline{12}$	4
$\underline{0}$	

402 =

- **Why do we like the standard algorithm?** Because it is quick, not too much to write down, and it works every time.
- **Why do we like the dots and boxes method?** Because it is easy to understand. (And drawing dots and boxes is kind of fun!)

We saw that 402 is evenly divisible by 3:  $402 \div 3 = 134$ . This means that 403, one more, shouldn't be divisible by three. It should be one dot too big. Do we see the extra dot if we try the dots and boxes method?

403 =

unexplode

=

stuck

Yes we do! We have one dot left at the end that can't be divided. We say that we have a *remainder* of one and some people like to write:  $403 \div 3 = 134 \text{ R}1$ .

$$\begin{array}{r} 134 \\ 3 \overline{)403} \\ \underline{3 \phantom{00}} \\ 10 \phantom{0} \\ \underline{9 \phantom{0}} \\ 13 \\ \underline{12} \\ 1 \end{array}$$

Let's try another one:  $263 \div 12$ . Here's what we have:

$$263 = \begin{array}{|c|c|c|} \hline \bullet\bullet & \bullet\bullet\bullet\bullet & \bullet\bullet\bullet \\ \hline \end{array}$$

And we are looking for groups like this:

$$12 = \begin{array}{|c|c|} \hline \bullet & \bullet\bullet \\ \hline \end{array}$$

Here goes!

$$263 = \begin{array}{|c|c|c|} \hline \bullet\bullet & \bullet\bullet\bullet\bullet & \bullet\bullet\bullet \\ \hline \end{array}$$

//    /

stuck

Unexploding won't help any further and we are indeed left with one remaining dot in the tens position and a dot in the ones position. This means we have a remainder of eleven.

$$263 \div 12 = 21 \text{ R}11.$$

### Think/Pair/Share.

- Use the dots and boxes method to compute each quotient:

$$5210 \div 4$$

$$4857 \div 23$$

$$31533 \div 101$$

- Now use the standard algorithm (an example is shown below) to compute each of the quotients above.

$$\begin{array}{r} 134 \\ 3 \overline{)403} \\ \underline{3} \phantom{0} \phantom{3} \\ 10 \phantom{3} \\ \underline{9} \phantom{3} \\ 13 \\ \underline{12} \\ 1 \end{array}$$

$$403 \div 3 = 134 \text{ R}1.$$

- Which method do you like better: dots and boxes or the standard algorithm method? Or does it depend on the problem you are doing?

**Think/Pair/Share.** Which is the *best* explanation for why this long division algorithm works? Think about your choice on your own. Then share your choice with a partner and see if you can agree on the best answer.

1. It works because you divide 37 into smaller parts of 4136 to make the problem easier to solve.
2. It works because you subtract multiples of powers of ten times 37 from 4136 until you have less than 37 left.
3. It works because if you multiply 111 by 37, and add in 29, you get 4136.
4. It works because you subtract 37s from 4136 until you have less than 37 left.

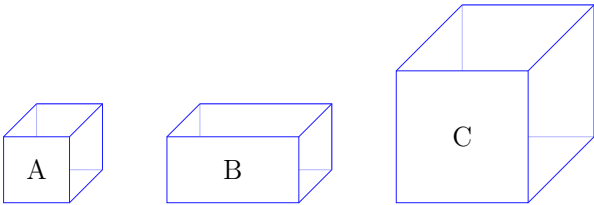
$$\begin{array}{r} 111 \text{ R } 29 \\ 37 \overline{)4136} \\ \underline{37} \phantom{00} \\ 43 \phantom{00} \\ \underline{37} \phantom{00} \\ 66 \phantom{00} \\ \underline{37} \phantom{00} \\ 29 \end{array}$$

# MEASUREMENT

Another way we often think about numbers is as abstract quantities that can be measured: length, area, and volume are all examples.

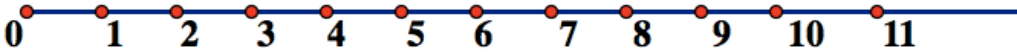
**Think/Pair/Share.** Answer the following questions about each picture:

- If  $A = 1$  unit, then what numbers would you assign to B and C? Why?
- If  $B = 1$  unit, then what numbers would you assign to A and C? Why?

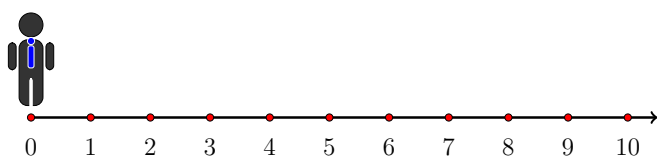


In a measurement model, you have to pick a *basic unit*. The basic unit is a quantity — length, area, or volume — that you assign to the number one. You can then assign numbers to other quantities based on how many of your basic unit fit inside.

For now, we'll focus on the quantity length, and we'll work with a number line where the basic unit is already marked off.



**3.1. Addition and Subtraction on the Number Line.** Imagine a person — we'll call him Zed — who can stand on the number line. We'll say that the distance Zed walks when he takes a step is exactly one unit.



When Zed wants to add or subtract with whole numbers on the number line, he always starts at 0 and faces the positive direction (towards 1). Then what he does depends on the calculation.

If Zed wants to *add* two numbers, he walks forward (to the right of the number line) however many steps are indicated by the first number (the first *addend*). Then he walks forward (to your right on the number line) the number of steps indicated by the second number (the second *addend*). Where he lands is the *sum* of the two numbers.

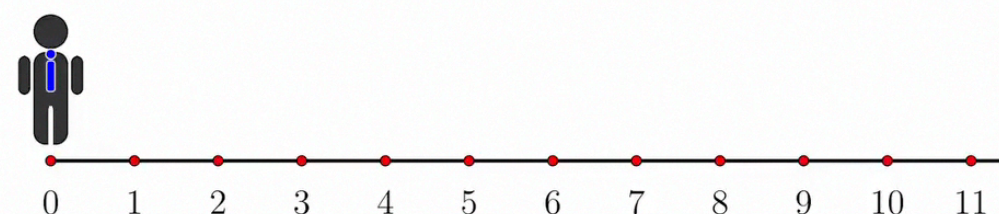
*Example 3.1* ( $3 + 4$ ). If Zed wants to add  $3 + 4$ , he starts at 0 and faces towards the positive numbers. He walks forward 3 steps, then he walks forward 4 more steps.

Zed ends at the number 7, so the sum of 3 and 4 is 7.  $3 + 4 = 7$ . (But you knew that of course! The point right now is to make sense of the *number line model*.)

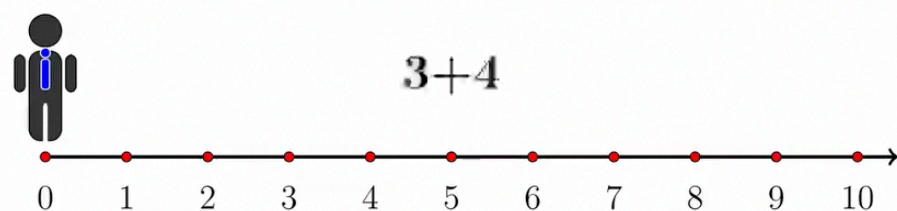
When Zed wants to *subtract* two numbers, he he walks forward (to the right on the number line) however many steps are indicated by the first number (the *minuend*). Then he walks *backwards* (to the left on the number line) the number of steps indicated by the second number (the *subtrahend*). Where he lands is the *difference* of the two numbers.

*Example 3.2* ( $11 - 3$ ). If Zed wants to subtract  $11 - 3$ , he starts at 0 and faces the positive numbers (the right side of the number line). He walks forward 11 steps on the number line, then he walks backwards 3 steps.

**Movie 7** *Example 3.2* ( $11 - 3$ )



**Movie 6** *Example 3.1* ( $3 + 4$ )





Zed ends at the number 8, so the difference of 11 and 3 is 8.  
 $11 - 3 = 8$ . (But you knew that!)

### Think/Pair/Share.

- Work out each of these exercises on a number line. You can actually pace it out on a life-sized number line or draw a picture:

$$4 + 5 \qquad 6 + 9 \qquad 10 - 7 \qquad 8 - 1$$

- Why does it make sense to walk forward for addition and walk backwards for subtraction? In what way is this the same as “combining” for addition and “take away” for subtraction?

- What happens if you do these subtraction problems on a number line? Explain your answers.

$$6 - 9 \qquad 1 - 7 \qquad 4 - 11 \qquad 0 - 1$$

- Could you do the subtraction problems above with the dots and boxes model?

### 3.2. Multiplication and Division on the Number Line.

Since multiplication is really repeated addition, we can adapt our addition model to become a multiplication model as well. Let’s think about  $3 \times 4$ . This means to add four to itself three times (that’s simply the definition of multiplication!):

$$3 \times 4 = 4 + 4 + 4.$$

So to multiply on the number line, we do the process for addition several times.

To multiply two numbers, Zed starts at 0 as always, and he faces the positive direction. He walks forward the number of steps given by the second number (the second *factor*). He repeats that process the number of times given by the first number (the first *factor*). Where he lands is the *product* of the two numbers.

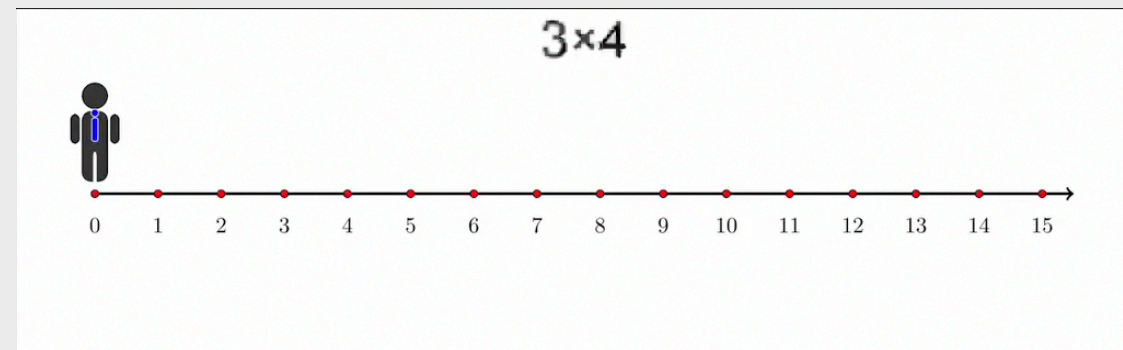
*Example 3.3* ( $3 \times 4$ ). If Zed wants to multiply  $3 \times 4$ , he can think of it this way:

$$\begin{array}{ccc} & 3 & \times & 4 \\ & \swarrow & & \searrow \\ \text{how many times to repeat it} & & & \text{how many steps to take forward} \end{array}$$

Zed starts at 0, facing the positive direction. The he repeats this three times: take four steps forward.



**Movie 8** Example 3.3 ( $3 \times 4$ )



He ends at the number 12, so the product of 3 and 4 is 12. That is,  $3 \times 4 = 12$ .

Remember our quotative model of division: One way to interpret  $15 \div 5$  is:

“How many groups of 5 fit into 15?”

Thinking on the number line, we can ask it this way:

“Zed takes 5 steps at a time. If Zed lands at the number 15, how many times did he take 5 steps?”

To calculate a division problem on the number line, Zed starts at 0, facing the positive direction. He walks forward the number of steps given by the second number (the *divisor*). He repeats that process until he lands at the first number (the *dividend*). The number of times he repeated the process gives the *quotient* of the two numbers.

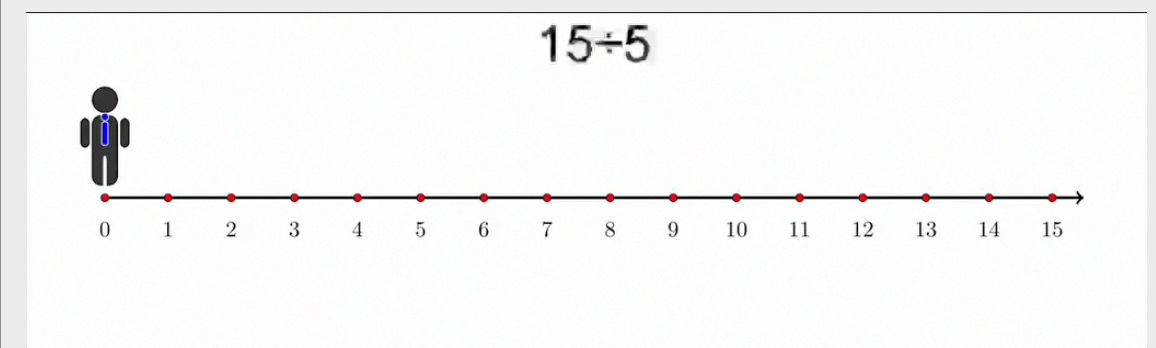
Example 3.4 ( $15 \div 5$ ). If Zed wants to compute  $15 \div 5$ , he can think of it this way:

$15 \div 5$   
where he wants to land      how many steps he takes at a time

He starts at 0, facing the positive direction.

- Zed takes 5 steps forward. He is now at 5, not 15. So he needs to repeat the process.
- Zed takes 5 steps forward again. He is now at 10, not 15. So he needs to repeat the process.
- Zed takes 5 more steps forward. He is at 15, so he stops.

**Movie 9** Example 3.4 ( $15 \div 5$ )



Since he repeated the process three times, we see there are 3 groups of 5 in 15. So the quotient of 15 and 5 is 3. That is,  $15 \div 5 = 3$ .

### Think/Pair/Share.

- Work out each of these exercises on a number line. You can actually pace it out on a life-sized number line or draw a picture:

$$2 \times 5$$

$$7 \times 1$$

$$10 \div 2$$

$$6 \div 1$$

- Can you think of a way to interpret these multiplication problems on a number line? Explain your ideas.

$$4 \times 0$$

$$0 \times 5$$

$$3 \times (-2)$$

$$2 \times (-1)$$

- What happens if you try to solve these division problems on a number line? Can you do it? Explain your ideas.

$$0 \div 2$$

$$0 \div 10$$

$$3 \div 0$$

$$5 \div 0$$

**3.3. Area Model for Multiplication.** So far we have focused on a linear measurement model, using the number line. But there's another common way to think about multiplication: using area.

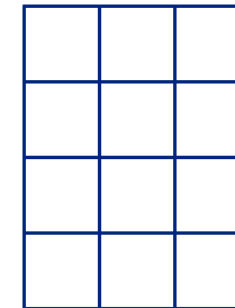
For example, suppose our basic unit is one square:



We can picture  $4 \times 3$  as 4 groups, with 3 squares in each group, all lined up:

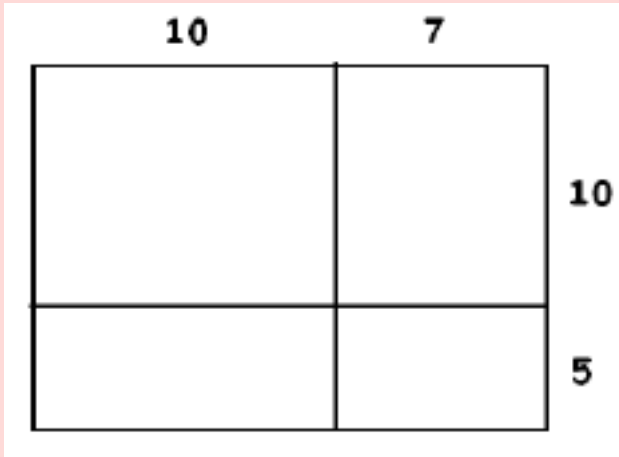


But we can also picture them stacked up instead of lined up. We would have 4 rows, with 3 squares in each row, like this:



So we can think about  $4 \times 3$  as a rectangle that has length 3 and width 4. The product, 12, is the total number of squares in that rectangle. (That is also the area of the rectangle, since each square was one unit!)

**Think/Pair/Share.** Vera drew this picture as a model for  $15 \times 17$ . Use her picture to help you compute  $15 \times 17$ . Explain your work.



**Problem 7.** Draw pictures like Vera’s for each of these multiplication exercises. Use your pictures to find the products without using a calculator or the standard algorithm.

$23 \times 37$ 
 $8 \times 43$ 
 $371 \times 42$

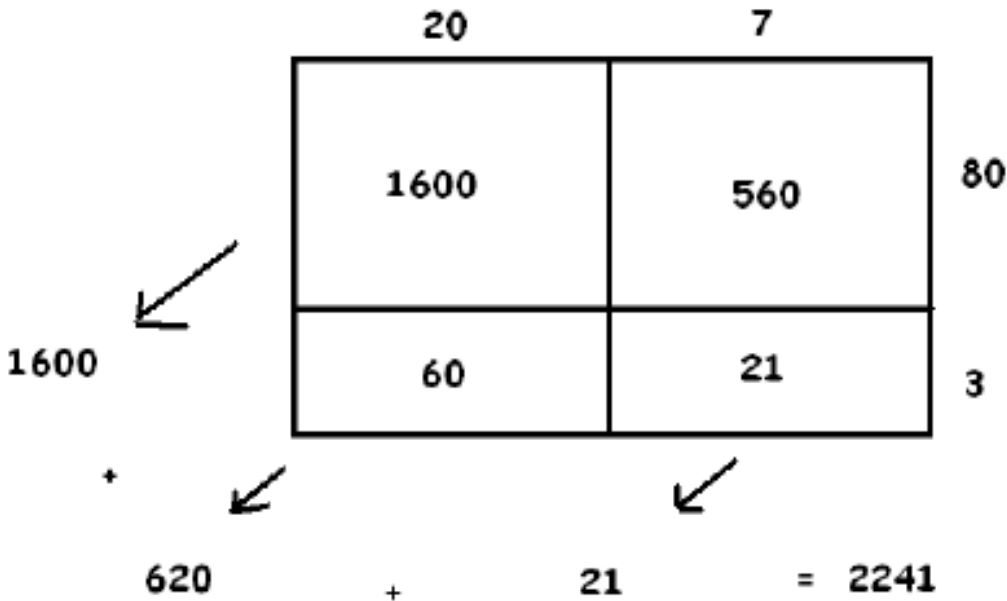
**3.4. The Standard Algorithm for Multiplication.** How were you taught to compute  $83 \times 27$  in school? Were you taught to write something like the following?

$$\begin{array}{r} 83 \\ \times 27 \\ \hline 21 \\ 56 \\ 6 \\ 16 \\ \hline 2241 \end{array}$$

Or maybe you were taught to put in the extra zeros rather than leaving them out?

$$\begin{array}{r} 83 \\ \times 27 \\ \hline 21 \\ 560 \\ 60 \\ 1600 \\ \hline 2241 \end{array}$$

This is really no different than drawing the rectangle and using Vera’s picture for calculating!



**Think/Pair/Share.**

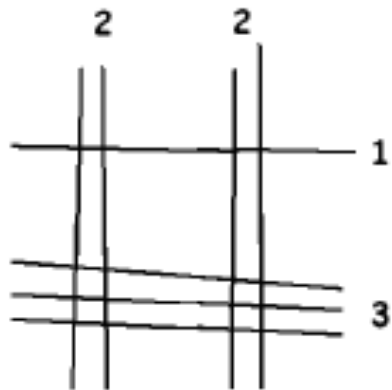
- Use the example above to explain why Vera’s rectangle method and the standard algorithm are really the same.
- Calculate the products below using both methods. Explain where you’re computing the same pieces in each algorithm.

$23 \times 14$

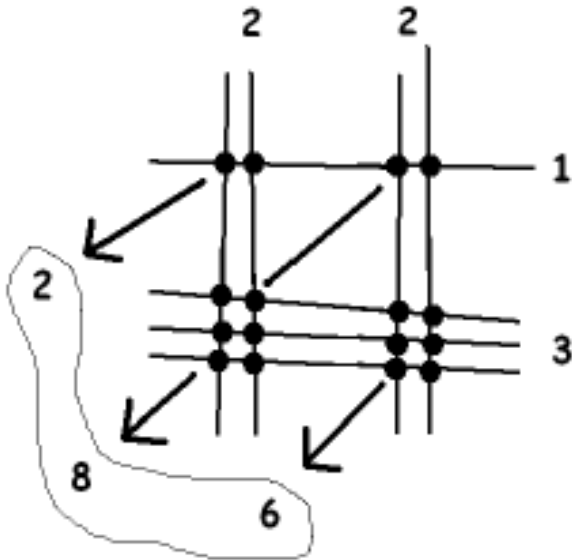
$106 \times 21$

$213 \times 31$

**Problem 8** (Lines and Intersections). Here’s an unusual way to perform multiplication. To compute  $22 \times 13$ , for example, draw two sets of vertical lines, the left set containing two lines and the right set two lines (for the digits in 22) and two sets of horizontal lines, the upper set containing one line and the lower set three (for the digits in 13).

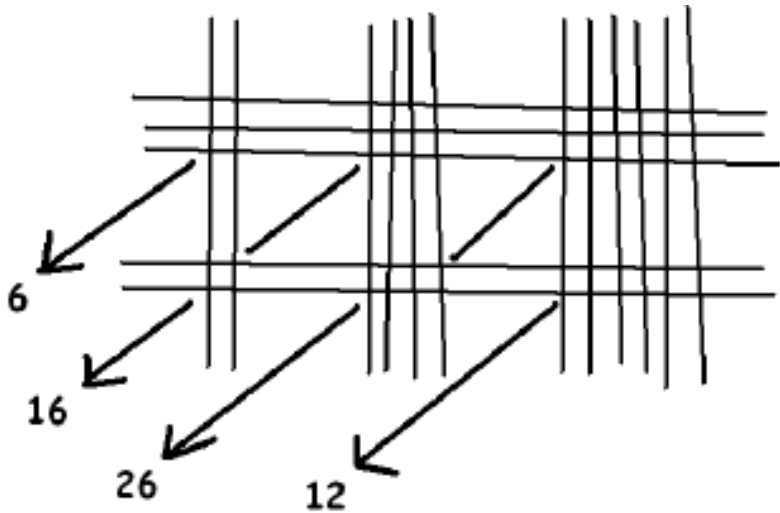


There are four sets of intersection points. Count the number of intersections in each and add the results diagonally as shown:



The answer 286 appears!

There is one possible glitch as illustrated by the computation  $246 \times 32$ :



Although the answer 6 thousands, 16 hundreds, 26 tens, and 12 ones is absolutely correct, one needs to carry digits and translate this as 7, 872.

**(Problem 8 continued).**

- (a) Compute  $131 \times 122$  via this method. Check your answer using another method.
- (b) Compute  $15 \times 1332$  via this method. Check your answer using another method.
- (c) Can you adapt the method to compute  $102 \times 3054$ ? (Why is some adaptation necessary?)
- (d) Why does the method work in general?

**Problem 9.** (Lattice Multiplication). In the 1500s in England, students were taught to compute multiplication using following galley method, now more commonly known as the *lattice method*.

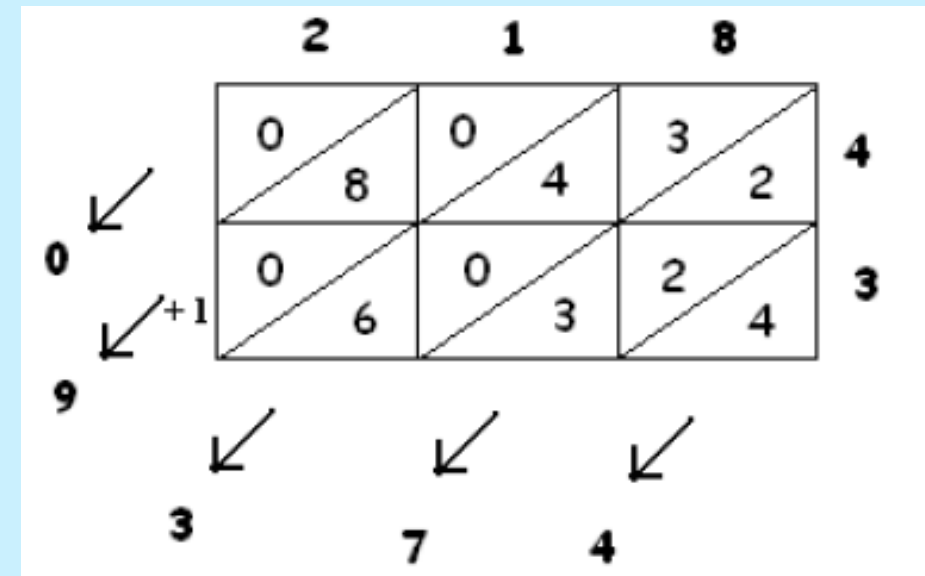
To multiply 43 and 218, for example, draw a  $2 \times 3$  grid of squares. Write the digits of the first number along the right side of the grid and the digits of the second number along the top.

Divide each cell of the grid diagonally and write in the product of the column digit and row digit of that cell, separating the tens from the units across the diagonal of that cell. (If the product is a one digit answer, place a 0 in the tens place.)

**(Problem 9 continued).**

Add the entries in each diagonal, carrying tens digits over to the next diagonal if necessary, to see the final answer. In our example, we have  $218 \times 43 = 9374$ .

Here's a picture of Lattice Multiplication:



- (a) Compute  $5763 \times 345$  via the lattice method.
- (b) Explain why the lattice method is really the standard algorithm in disguise.
- (c) What is the specific function of the diagonal lines in the grid?



# OPERATIONS

So far, you have seen a couple of different *models* for the operations: addition, subtraction, multiplication, and division. But we haven't talked much about the operations themselves — how they relate to each other, what properties they have that make computing easier, and how some special numbers behave. There's lots to think about!

The goal in this section is to use the models to understand why the operations behave according to the rules you learned back in elementary school. We're going to keep asking ourselves "Why does it work this way?"

**Think/Pair/Share.** Each of these models lends itself to thinking about the operation in a slightly different way. Before we really dig in to thinking about the operations, discuss with a partner:

- Of the models we discussed so far, do you prefer one of them?
- How well do the models we discussed match up with how you usually think about whole numbers and their operations?
- Which models are useful for computing? Why?
- Which models do you think will be useful for explaining how the operations work? Why?

**4.1. Relationships Between the Operations.** We defined addition as combining two quantities and subtraction as "taking away." But in fact, these two operations are intimately tied together. These two questions are exactly the same:

$$27 - 13 = \underline{\quad\quad\quad} \qquad 27 = 13 + \underline{\quad\quad\quad}.$$

More generally, for any three whole numbers  $a$ ,  $b$ , and  $c$ , these two equations express the same fact. (So either both equations are true or both are false. Which is the case depends on the values you choose for  $a$ ,  $b$ , and  $c$ !)

$$c - b = a \qquad c = a + b.$$

In other words, we can think of every subtraction problem as a "missing addend" addition problem. Try it out!

**Problem 10.** Here is a strange addition table. Use it to solve the following problems. Justify your answers. Important: Don't try to assign numbers to  $A$ ,  $B$ , and  $C$ . Solve the problems just using what you know about the operations!



+	<i>A</i>	<i>B</i>	<i>C</i>
<i>A</i>	<i>C</i>	<i>A</i>	<i>B</i>
<i>B</i>	<i>A</i>	<i>B</i>	<i>C</i>
<i>C</i>	<i>B</i>	<i>C</i>	<i>A</i>

$A + C$      $B + C$      $A - C$      $C - A$      $A - A$      $B - C$

**Think/Pair/Share.** Discuss your answers with a partner. How does an addition table help you solve subtraction problems?

We defined multiplication as repeated addition and division as forming groups of equal size. But in fact, these two operations are also tied together. These two questions are exactly the same:

$$27 \div 3 = \underline{\hspace{2cm}} \qquad 27 = \underline{\hspace{2cm}} \times 3.$$

More generally, for any three whole numbers  $a$ ,  $b$ , and  $c$ , these two equations express the same fact. (So either both equations are true or both are false. Which is the case depends on the values you choose for  $a$ ,  $b$ , and  $c$ !)

$$c \div b = a \qquad c = a \cdot b.$$

In other words, we can think of every division problem as a “missing factor” multiplication problem. Try it out!

**Problem 11.** Rewrite each of these division questions as a “missing factor” multiplication question. Which ones can you solve and which can you not solve? Explain your answers.

$$9 \div 3 \qquad 100 \div 25 \qquad 0 \div 3 \qquad 9 \div 0 \qquad 0 \div 0$$

**Problem 12.** Here’s a multiplication table.

$\times$	<i>A</i>	<i>B</i>	<i>C</i>	<i>D</i>	<i>E</i>
<i>A</i>	<i>A</i>	<i>A</i>	<i>A</i>	<i>A</i>	<i>A</i>
<i>B</i>	<i>A</i>	<i>B</i>	<i>C</i>	<i>D</i>	<i>E</i>
<i>C</i>	<i>A</i>	<i>C</i>	<i>E</i>	<i>B</i>	<i>D</i>
<i>D</i>	<i>A</i>	<i>D</i>	<i>B</i>	<i>E</i>	<i>C</i>
<i>E</i>	<i>A</i>	<i>E</i>	<i>D</i>	<i>C</i>	<i>B</i>

- Use the table to solve the problems below. Justify your answers. Important: Don’t try to assign numbers to the letters. Solve the problems just using what you know about the operations!

$$C \times D \qquad C \times A \qquad A \times A \qquad C \div D \qquad D \div C \qquad D \div E$$

- Can you use the table to solve these problems? Explain your answers.

$$D^2 \qquad C^3 \qquad A \div C \qquad A \div D \qquad C \div A \qquad D \div A \qquad A \div A$$

**Think/Pair/Share.** Discuss your answers to problems 11 and 12 with a partner. How does a multiplication table help you solve division (and exponentiation) problems?

Throughout this course, our focus is on explanation and justification. As teachers, you need to know what is true in mathematics, but you also need to know *why* it is true. And you will need lots of ways to explain *why*, since different explanations will make sense to different students.

It's very important not to confuse the fact (or rule) with the reason it is true! Here are some examples:

*Example 4.1* (Explaining the Connection Between Addition and Subtraction).

**Arithmetic Fact:**  $a + b = c$  and  $c - b = a$  are the same mathematical fact.

**Why It's True, Explanation 1:** First we'll use the definition of the operations.

Suppose we know  $c - b = a$  is true. Subtraction means "take away." So

$$c - b = a$$

means we start with quantity  $c$  and take away quantity  $b$ , and we end up with quantity  $a$ . Start with this equation, and imagine adding quantity  $b$  to both sides.

On the left, that means we started with quantity  $c$ , took away  $b$  things, and then put those  $b$  things right back! Since we took away some quantity and then added back the exact same quantity, there's no overall change. We're left with quantity  $c$ .

On the right, we would be combining (adding) quantity  $a$  with quantity  $b$ . So we end up with:  $c = a + b$ .

On the other hand, suppose we know the equation  $a + b = c$  is true. Imagine taking away (subtracting) quantity  $b$  from both sides of this equation:  $a + b = c$ .

On the left, we started with  $a$  things and combined that with  $b$  things, but then we immediately take away those  $b$  things. So we're left with just our original quantity of  $a$ .

On the right, we start with quantity  $c$  and take away  $b$  things. That's the very definition of  $c - b$ . So we have the equation:  $a = c - b$ .

**Movie 10** *Example 4.1*

$$c - b = a$$

**Why It's True, Explanation 2:** Let's use the measurement model to come up with another explanation.

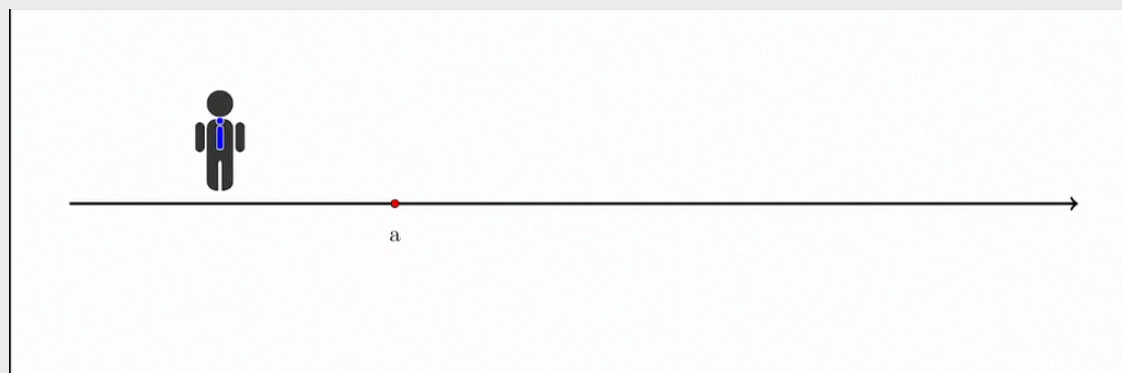
The equation  $a + b = c$  means Zed starts at 0, walks forward  $a$  steps, and then walks forward  $b$  steps, and he ends at  $c$ .

If Zed wants to compute  $c - b$ , he starts at 0, walks forward  $c$  steps, and then walks backwards  $b$  steps. But we know that to walk forward  $c$  steps, he can first walk forward  $a$  steps and then walk forward  $b$  steps. So Zed can compute  $c - b$  this way:

- Start at 0.
- Walk forward  $a$  steps.
- Walk forward  $b$  steps. (Now at  $c$ , since  $a + b = c$ .)
- Walk backwards  $b$  steps.

The last two steps cancel each other out, so Zed lands back at  $a$ . That means  $c - b = a$ .

**Movie 11 Zed**



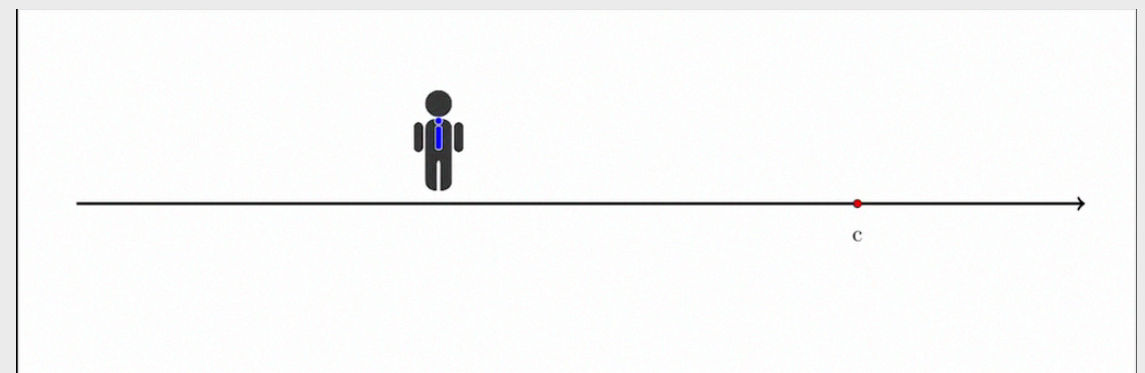
On the other hand, the equation  $c - b = a$  means that Zed starts at 0, walks forward  $c$  steps, then walks backwards  $b$  steps, and he ends up at  $a$ .

If Zed wants to compute  $a + b$ , he starts at 0, walks forward  $a$  steps, and then walks forwards  $b$  additional steps. But we know that to walk forward  $a$  steps, he can first walk forward  $c$  steps and then walk backwards  $b$  steps. So Zed can compute  $a + b$  this way:

- Start at 0.
- Walk forward  $c$  steps.
- Walk backwards  $b$  steps. (Now at  $a$ , since  $c - b = a$ .)
- Walk forward  $b$  steps.

The last two steps cancel each other out, so Zed lands back at  $c$ . That means  $a + b = c$ .

**Movie 12 Zed**



### Think/Pair/Share.

- Read over the two explanations in the example above. Do you think either one is more clear than the other?
- Why is this *not* a good explanation of the mathematical fact?

“I can check that this is true! For example,  $2+3 = 5$  and  $5-3 = 2$ . And  $3 + 7 = 10$  and  $10 - 7 = 3$ . It works for whatever numbers you try.”

- Use either the definition of the operations or the measurement model to explain the connection between multiplication and division:

$c \div b = a$  is the same fact as  $c = a \times b$ .

**4.2. Properties of Addition and Subtraction.** You probably know several properties of addition, but you may never have stopped to wonder: *Why is that true?!* Now’s your chance! In this section, you’ll use the definition of the operations of addition and subtraction and the models you’ve learned to explain why these properties are always true.

Here are the three properties you’ll think about:

- Addition of whole numbers is *commutative*.
- Addition of whole numbers is *associative*.
- The number 0 is an *identity* for addition of whole numbers.

For each of the properties, we don’t want to confuse these three ideas:

- what the property is called and what it means (the definition),
- some examples that *demonstrate* the property, and
- an explanation for *why* the property holds.

Notice that *examples* and *explanations* are not the same! These properties are all universal statements — statements of the form “for all,” “every time,” “always,” etc. That means that to show they are true, you either have to check every case or find a *reason why* it must be so.

Since there are infinitely many whole numbers, it’s impossible to check every case. You’d never finish! Our only hope is to look for *general explanations*.

*Example 4.2* (Commutativity). We’ll work out the explanation for the first of these facts, and you will work on the others.

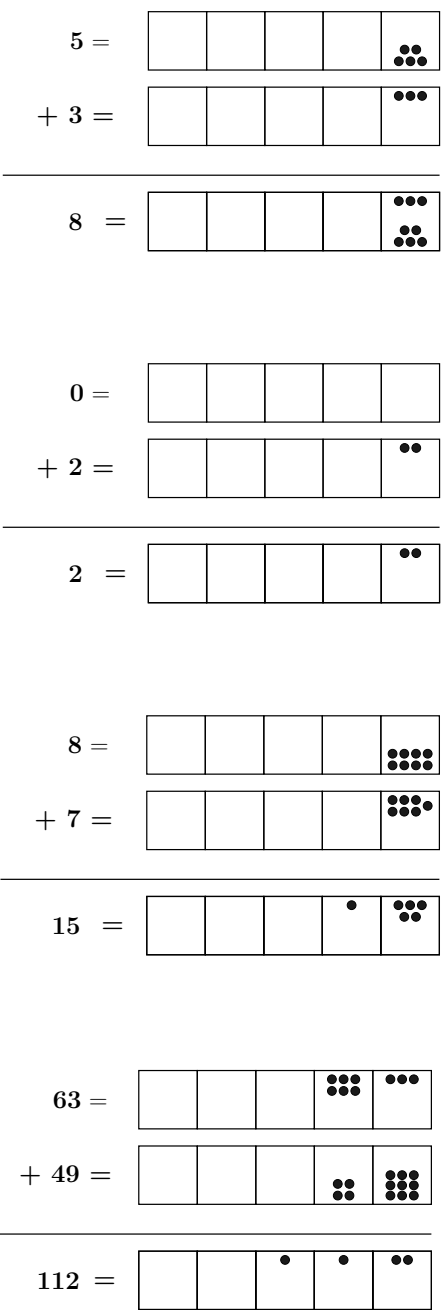
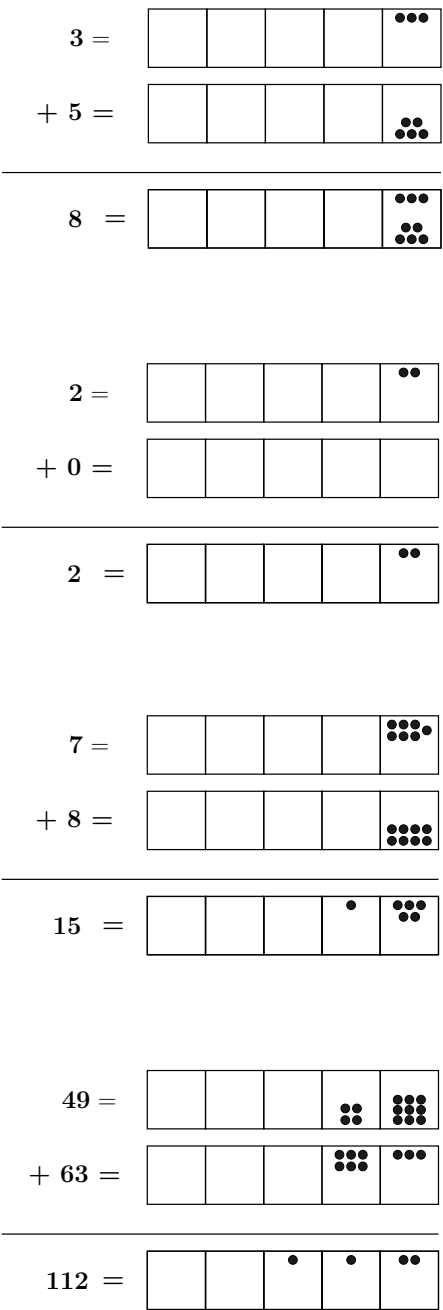
**Property:** Addition of whole numbers is commutative.

**What it Means (words):** When I add two whole numbers, the order I add them doesn’t affect the sum.

**What it Means (symbols):** For any two whole numbers  $a$  and  $b$ ,

$$a + b = b + a.$$

Examples:



Now we need a *justification*. *Why* is addition of whole numbers commutative?

**Why It's True, Explanation 1:** Let's think about addition as combining two quantities of dots.

- To add  $a + b$ , we take  $a$  dots and  $b$  dots, and we combine them in a box. To keep things straight, let's imagine the  $a$  dots are colored red and the  $b$  dots are colored blue. So in the box we have  $a$  red dots,  $b$  blue dots and  $a + b$  total dots.
- To add  $b+a$ , let's take  $b$  blue dots and  $a$  red dots, and put them all together in a box. We have  $b$  blue dots,  $a$  red dots and  $b + a$  total dots.
- But the total number of dots are the same in the two boxes! How do we know that? Well, there are  $a$  red dots in each box, so we can match them up. There are  $b$  blue dots in each box, so we can match them up. That's it! If we can match up the dots one-for-one, there must be the same number of them!
- That means  $a + b = b + a$ .

**Why It's True, Explanation 2:** We can also use the measurement model to explain why  $a + b = b + a$  no matter what numbers we choose for  $a$  and  $b$ . Imagine taking a segment of length  $a$  and combining it linearly with a segment of length  $b$ . That's how we get a length of  $a + b$ .





But if we just rotate that segment so it's upside down, we see that we have a segment of length  $b$  combined with a segment of length  $a$ , which makes a length of  $b + a$ .



But of course it's the same segment! We just turned it upside down! So the lengths must be the same. That is,  $a + b = b + a$ .

**Problem 13** (Addition is Associative). Your turn! You'll answer the question, "Why is addition of whole numbers associative?"

**Property:** Addition of whole numbers is associative.

**What it Means (words):** When I add three whole numbers in a given order, the way I group them (to add two at a time) doesn't affect the sum.

**What it Means (symbols):** For any three whole numbers  $a$ ,  $b$ , and  $c$ ,

$$(a + b) + c = a + (b + c).$$

- (a) Come up with at least three *examples* to demonstrate associativity of addition.
- (b) Use our models of addition to come up with an *explanation*. Why does associativity hold in *every case*?

**Problem 14** (Identity for Addition). Why is the number 0 an identity for addition?

**Property:** The number 0 is an *identity* for addition of whole numbers.

**What it Means (words):** When I add any whole number to 0 (in either order), the sum is the very same whole number I added to 0.

**What it Means (symbols):** For any whole numbers  $n$ ,

$$n + 0 = n \quad \text{and} \quad 0 + n = n.$$

- (a) Come up with at least three *examples* to demonstrate that 0 is an identity for addition.
- (b) Use our models of addition to come up with an *explanation*. Why does this property of 0 hold in *every possible case*?

Since addition and subtraction are so closely linked, it's natural to wonder if subtraction has some of the same properties as addition, like commutativity and associativity.

*Example 4.3.* Justin asked if the operation of subtraction is commutative. That would mean that the difference of two whole numbers doesn't depend on the order in which you subtract them. In symbols: *for every choice* of whole numbers  $a$  and  $b$  we would have  $a - b = b - a$ .

Jared says that subtraction is *not* commutative since  $4 - 3 = 1$ , but  $3 - 4 \neq 1$ . (In fact,  $3 - 4 = -1$ .)

Since the statement “subtraction is commutative” is a *universal statement*, one counterexample is enough to show it’s not true. So Jared’s counterexample lets us say with confidence:

Subtraction is **not** commutative.

**Think/Pair/Share.** Can you find any examples of whole numbers  $a$  and  $b$  where  $a - b = b - a$  is true? Explain your answer.

**Problem 15.** Lyle asked if the operation of subtraction is associative.

- (a) State what it would mean for subtraction to be associative. You should use words and symbols.
- (b) What would you say to Lyle? Decide if subtraction is associative or not. Carefully explain how you made your decision and *how you know you’re right*.

**Problem 16.** Jess asked if the number 0 is an identity for subtraction.

- (a) State what it would mean for 0 to be an identity for subtraction. You should use words and symbols.
- (b) What would you say to Jess? Decide if 0 is an identity for subtraction or not. Carefully explain how you made your decision and *how you know you’re right*.

**4.3. Properties of Multiplication and Division.** Now we’re going to turn our attention to familiar properties of multiplication and division, with the focus still on explaining why these properties are always true.

Here are the four properties you’ll think about:

- Multiplication of whole numbers is *commutative*.
- Multiplication of whole numbers is *associative*.
- Multiplication of whole numbers *distributes over addition*
- The number 1 is an *identity* for multiplication of whole numbers.

For each of the properties, remember to keep straight:

- what the property is called and what it means (the definition),
- some examples that *demonstrate* the property, and
- an explanation for *why* the property holds.

Once again, it’s important to distinguish between *examples* and *explanations*. They are not the same! Since there are infinitely many whole numbers, it’s impossible to check every case, so examples will never be enough to explain why these properties hold. You have to figure out *reasons* for these properties to hold, based on what you know about the operations.



*Example 3.4 (Identity).* We'll work out the explanation for the last of these facts, and you will work on the others.

**Property:** The number 1 is an identity for multiplication of whole numbers.

**What it Means (words):** When I multiply a number by 1 (in either order), the product is the same as that number I multiplied by 1.

**What it Means (symbols):** For any whole numbers  $m$ ,

$$m \times 1 = m \quad \text{and} \quad 1 \times m = m.$$

**Examples:**  $1 \times 5 = 5$ ,  $19 \times 1 = 19$ , and  $1 \times 1 = 1$ .

*Why* does the number 1 act this way with multiplication?

**Why It's True, Explanation 1:** Let's think first about the definition of multiplication as repeated addition:

- $m \times 1$  means to add the number one to itself  $m$  times:

$$\frac{1 + 1 + \dots + 1}{m \text{ ones}}$$

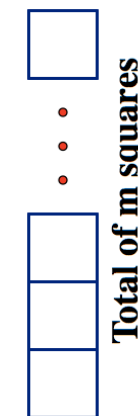
So we see that  $m \times 1 = m$  for any whole number  $m$ .

- On the other hand,  $1 \times m$  means to add the number  $m$  to itself just one time. This might seem a little confusing, since there's no actual addition going on. But if this makes any sense at all, the answer must be  $m$ . So  $1 \times m = m$  also.

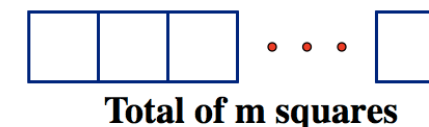
**Why It's True, Explanation 2:** We can also use the number line model to create a justification. If Zed calculates  $1 \times m$ , he will start at 0 and face the positive direction. He will then take  $m$  steps forward, and he will do it just one time. So he lands at  $m$ , which means  $1 \times m = m$ .

If Zed calculates  $m \times 1$ , he starts at 0 and faces the positive direction. Then he takes one step forward, and he repeats that  $m$  times. So he lands at  $m$ . We see that  $m \times 1 = m$ .

**Why It's True, Explanation 3:** In the area model,  $m \times 1$  represents  $m$  rows with one square in each row. That makes a total of  $m$  squares. So  $m \times 1 = m$ .



Similarly,  $1 \times m$  represents one row of  $m$  squares. That's also a total of  $m$  squares. So  $1 \times m = m$ .



**Think/Pair/Share.** The example presented several different explanations. Do you think one is more convincing than the others? Or more clear and easier to understand?

Now it's your turn to come up with some explanations.

**Problem 17** (Multiplication is Commutative). Why is multiplication of whole numbers commutative?

**Property:** Multiplication whole numbers is commutative.

**What it Means (words):** When I multiply two whole numbers, switching the order in which I multiply them does not affect the product.

**What it Means (symbols):** For any two whole numbers  $a$  and  $b$ ,

$$a \cdot b = b \cdot a.$$

- (a) Come up with at least three *examples* to demonstrate the commutativity of multiplication.
- (b) Use our models of multiplication to come up with an *explanation*. Why does commutativity hold in *every case*?

**Problem 18** (Multiplication is Associative). Why is multiplication of whole numbers associative?

**Property:** Multiplication of whole numbers is associative.

**What it Means (words):** When I multiply three whole numbers in a given order, the way I group them (to multiply two at a time) doesn't affect the product.

**What it Means (symbols):** For any three whole numbers  $a$ ,  $b$ , and  $c$ ,

$$(a \cdot b) \cdot c = a \cdot (b \cdot c).$$

- (a) Come up with at least three *examples* to demonstrate the associativity of multiplication.
- (b) Use our models of multiplication to come up with an *explanation*. Why does associativity hold in *every case*?

**Property:** Multiplication distributes over addition.

**What it means:** The distributive law for multiplication over addition is a little hard to state in words, so we'll jump straight to the symbols. For any three whole numbers  $x$ ,  $y$ , and  $z$ :

$$x \cdot (y + z) = x \cdot y + x \cdot z.$$

**Examples:**

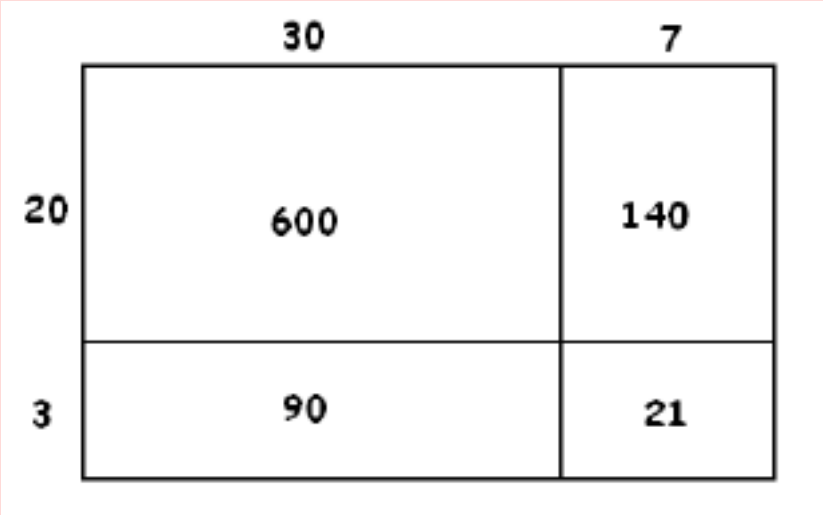
$8 \cdot (23) = 8 \cdot (20 + 3) = 8 \cdot 20 + 8 \cdot 3 = 160 + 24 = 184$

$5 \cdot (108) = 5 \cdot (100 + 8) = 5 \cdot 100 + 5 \cdot 8 = 500 + 40 = 540$

**Think/Pair/Share.** We actually did calculations very much like the examples above, when we looked at the area model for multiplication.

- Draw a picture for each example above.

This picture shows our method for multiplying  $23 \times 37$ .



We can also write the calculation this way:

$23 \cdot 37 = (20 + 3)(30 + 7)$

$= 20 \cdot 30 + 20 \cdot 7 + 3 \cdot 30 + 3 \cdot 7$

$= 600 + 140 + 90 + 21$

$= 851$

- Use the distributive rule (twice!) to explain why this calculation works.

**Problem 19.** Which of the following pictures best represents the distributive law in the equation  $3 \cdot (2 + 4) = 3 \cdot 2 + 3 \cdot 4$ ? Explain your choice.

a)

b)

c)

d)

e)

**Problem 20.** Use the distributive law to easily compute each of these in your head (no calculators!). Explain your solution.

$45 \times 11$        $63 \times 101$        $172 \times 1001$

**Think/Pair/Share.** Use one of our models for multiplication and addition to explain why the distributive rule works every time.

It's natural to wonder which, if any, of these properties also hold for division (since you know that the operations of multiplication and division are connected).

*Example 3.5.* If division were associative, then for any choice of three whole numbers  $a$ ,  $b$ , and  $c$ , we would have

$$a \div (b \div c) = (a \div b) \div c.$$

Remember, the parentheses tell you which two numbers to divide first.

Let's try the example  $a = 9$ ,  $b = 3$ , and  $c = 1$ . Then

$$a \div (b \div c) = 9 \div (3 \div 1) = 9 \div 3 = 3 \quad \text{and}$$

$$(a \div b) \div c = (9 \div 3) \div 1 = 3 \div 1 = 3.$$

So is it true? Is division associative? Well, we can't be sure. This is just one example. But "division is associative" is a *universal statement*. If it's true, it has to work for *every possible example*. Maybe we just stumbled on a good choice of numbers, but it won't always work.

Let's keep looking. Try  $a = 16$ ,  $b = 4$ , and  $c = 2$ .

$$a \div (b \div c) = 16 \div (4 \div 2) = 16 \div 2 = 8 \text{ and}$$

$$(a \div b) \div c = (16 \div 4) \div 2 = 4 \div 2 = 2.$$

That's all we need! A single counterexample lets us conclude:

Division is **not** associative.

**Problem 21.** What about the other properties? It's your turn to decide!

- (a) State what it would mean for division to be commutative. You should use words and symbols.
- (b) Decide if division is commutative or not. Carefully explain how you made your decision and *how you know you're right*.
- (c) State what it would mean for division to distribute over addition. You definitely want to use symbols!
- (d) Decide if division distributes over addition or not. Carefully explain how you made your decision and *how you know you're right*.
- (e) State what it would mean for the number 1 to be an identity for division. You should use words and symbols.
- (f) Decide if 1 is an identity for division or not. Carefully explain how you made your decision and *how you know you're right*.

**Problem 22** (Zero property). You probably know another property of multiplication that hasn't been mentioned yet:

If I multiply any number times 0 (in either order), the product is 0. This is sometimes called the zero property of multiplication. Notice that the *zero property* is very different from the property of being an identity!

- (a) Write what the zero property means in symbols: For every whole number  $n$  . . .
- (b) Give at least three examples of the zero property for multiplication.
- (c) Use one of our models of multiplication to explain why the zero property holds.

**Think/Pair/Share** (Division by 0). Which of the following best explains why division by 0 is undefined? First decide for yourself. Then share your thoughts with a partner and see if you agree.

- (1) Division by 0 is undefined because you cannot do it.
- (2) Division by 0 is undefined because you cannot make 0 groups of something.
- (3) Division by 0 is undefined because there is no single number that, when multiplied by 0, gives the original number.
- (4) Division by 0 is undefined because every number divided by 0 equals 0.

**Think/Pair/Share** (More on division by 0).

- For each division problem below, turn it into a multiplication problem. Solve those problems if you can. If you can't, explain what is wrong.

$$5 \div 0 \quad 0 \div 5 \quad 7 \div 0 \quad 0 \div 7 \quad 0 \div 0$$

- Use your work to explain why we say that *division by 0 is undefined*.
- Use one of our models of division to explain why *division by 0 is undefined*.

**4.4. Four Fact Families.** In elementary school, students are often encouraged to memorize “four fact families,” for example:

$$2 + 3 = 5 \quad 5 - 3 = 2$$

$$3 + 2 = 5 \quad 5 - 2 = 3$$

Here's a different “four fact family”:

$$2 \cdot 3 = 6 \quad 6 \div 3 = 2$$

$$3 \cdot 2 = 6 \quad 6 \div 2 = 3$$

**Think/Pair/Share.**

- In what sense are these groups of equations “families”?
- Write down at least two more addition / subtraction four fact families.
- Use properties of addition and subtraction to explain *why* these four fact families are each really one fact.
- Write down at least two more multiplication / division four fact families.
- Use properties of multiplication and division to explain *why* these four fact families are each really one fact.

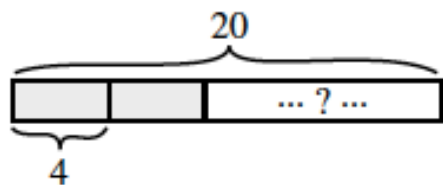
**Problem 23.**

(a) Here’s a true fact in base six:  $2_{\text{six}} + 3_{\text{six}} = 5_{\text{six}}$ . Write the rest of this four fact family.

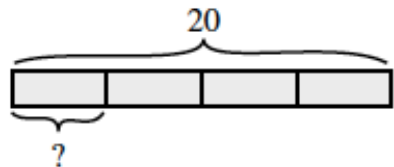
(b) Here’s a true fact in base six:  $11_{\text{six}} - 5_{\text{six}} = 2_{\text{six}}$ . Write the rest of this four fact family.

**4.5. Going Deeper with Division.** So far we’ve been thinking about division in what’s called the *quotative model*. In the quotative model, we want to make groups of equal size. We know the *size of the group*, and we ask *how many groups*. For example, we think of  $20 \div 4$  as:

How many groups of 4 are there in a group of 20?



Thinking about four fact families, however, we realize we can turn the question around a bit. We could think about the *partitive model* of division. In the partitive model, we want to make an equal number of groups. We know *how many groups*, and we ask the *size of the group*. In the partitive model, we think of  $20 \div 4$  as: 20 is 4 groups of what size?



When we know the original amount and the number of parts, we use partitive division to find the size of each part.

When we know the original amount and the size of each part, we use quotative division to find the number of parts.

Here are some examples in word problems:

PARTITIVE	QUOTATIVE
number of groups known find the number in each group	number in each group known find the number of groups
Sylvia makes \$26,000 per year. How much does she make weekly?	Sylvia makes \$650 weekly. Last year she made \$26,000. How many weeks did she work?
A movie theater made \$6450 in one night of ticket sales. 430 people purchased a ticket. How much does each ticket cost?	A movie theater made \$6450 in one night of ticket sales. Each ticket costs \$12.50. How many people purchased a ticket?



**Think/Pair/Share.** For each word problem below:

- Draw a picture to show what the problem is asking.
- Use your picture to help you decide if it is a quotative or a partitive division problem.
- Solve the problem using any method you like.

- (1) David made 36 cookies for the bake sale. He packaged the cookies in boxes of 9. How many boxes did he use?
- (2) David made 36 cookies to share with his friends at lunch. There were 12 people at his lunch table (including David). How many cookies did each person get?
- (3) Liz spent one summer hiking the Appalachian trail. She completed 1,380 miles of the trail and averaged 15 miles per day. How many days was she out hiking that summer?
- (4) On April 1, 2012, **Chase Norton** became the first person to hike the entire Ko'olau summit in a single trip. (True story!) It took him eight days to hike all 48 miles from start to finish. If he kept a steady pace, how many miles did he hike each day?

**Think/Pair/Share.** Write your own word problems: Write one partitive division problem and one quotative division problem. Choose your numbers carefully so that the answer works out nicely. Be sure to solve your problems!

Why think about these two models for division? You won't be teaching the words *partitive* and *quotative* to your students. But recognizing the two kinds of division problems (and being able to come up with examples of each) will make you a better teacher.

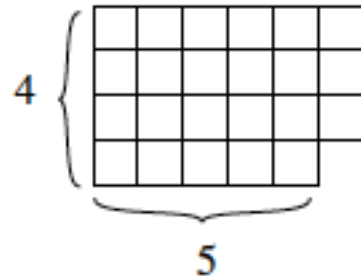
It's important that your students are exposed to both ways of thinking about division, and to problems of both types. Otherwise, they may think about division too narrowly and not really understand what's going on. If you understand the two kinds of problems, you can more easily diagnose and remedy students' difficulties.

Most of the division problems we've looked at so far have come out evenly, with no remainder. But of course, that doesn't always happen!

**Problem 24.** What is  $43 \div 4$ ?

- (a) Write a problem that uses the computation  $43 \div 4$  and gives 10 as the correct answer.
- (b) Write a problem that uses the computation  $43 \div 4$  and gives 11 as the correct answer.
- (c) Write a problem that uses the computation  $43 \div 4$  and gives 10.75 as the correct answer.

We can think about division with remainder in terms of some of our models for operations. For example, we can calculate that  $23 \div 4 = 5 \text{ R}3$ . We can picture it this way:



**Think/Pair/Share.**

- Explain how the picture above illustrates  $23 \div 4 = 5 \text{ R}3$ . Where do you see the remainder of 3 in the picture?
- Explain the connection between these two equations.

$$23 \div 4 = 5 \text{ R}3 \quad \text{and}$$

$$23 = 5 \cdot 4 + 3.$$

- How could you use the number line model to show the calculation  $23 \div 4 = 5 \text{ R}3$ ? What does a “remainder” look like in this model?
- Draw area models for each of these division problems. Find the quotient and remainder.

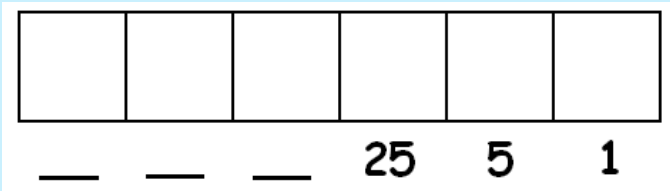
$$40 \div 12$$

$$59 \div 10$$

$$91 \div 16$$

# DIVISION EXPLORATIONS

**Problem 25** (Base 5 Division). Remember that base five numbers are in a  $1 \leftarrow 5$  dots-and-boxes system. What are the place values in the  $1 \leftarrow 5$  system? Fill in the blanks:



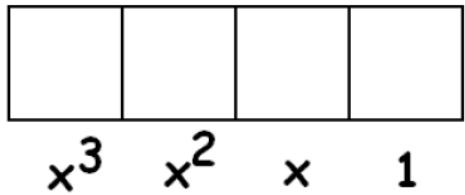
- Draw a dots-and-boxes picture of the number  $424_{\text{five}}$ .
- Draw a dots-and-boxes picture of the number  $11_{\text{five}}$ .
- Use the dots and boxes method to show that  $424_{\text{five}} \div 11_{\text{five}} = 34_{\text{five}}$ .
- Rewrite the division sentence  $424_{\text{five}} \div 11_{\text{five}} = 34_{\text{five}}$  in base 10, and check that it's correct.
- **Challenge!** Use dots-and-boxes to find  $2021_{\text{five}} \div 12_{\text{five}}$ .  
*Don't convert to base 10!*

**Problem 26** (Base. . .  $x$ ?!?!). Anu refuses to tell anyone if she is working in a  $1 \leftarrow 10$  system, or a  $1 \leftarrow 5$  system, or any other system. She makes everyone call it a  $1 \leftarrow x$  system but won't tell a soul what number she has in mind for  $x$ .

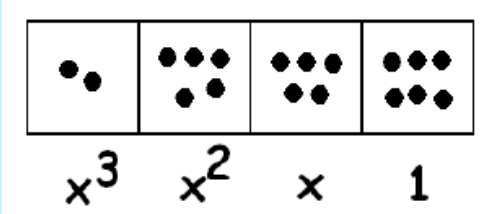
We know that boxes in a  $1 \leftarrow 10$  have values that are powers of ten: 1, 10, 100, 1000, 10000, . . .

And boxes in a  $1 \leftarrow 5$  system are powers of five: 1, 5, 25, 125, 625, . . .

So Anu's system, whatever it is, must be powers of  $x$ :

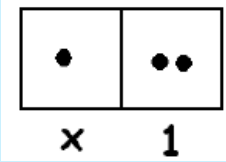


When Anu writes  $2556_x$  she must mean:



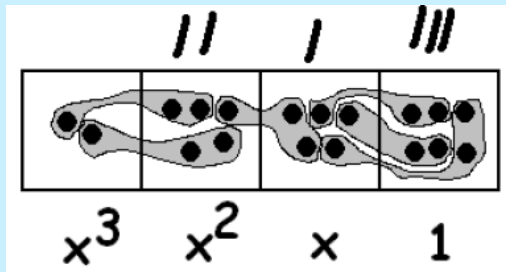
$$2x^3 + 5x^2 + 5x + 6$$

And when she writes  $12_x$  she means:



$$x + 2.$$

Anu decides to compute  $2556_x \div 12_x$ . She obtains:



$$(2x^3 + 5x^2 + 5x + 6) \div (x + 2) = 2x^2 + x + 3.$$

(a) Check Anu's division by computing

$$(x + 2)(2x^2 + x + 3)$$

Did it work?

(b) Use Anu's method to find  $(3x^2 + 7x + 2) \div (x + 2)$ .

(c) Use Anu's method to find

$$(2x^4 + 3x^3 + 5x^2 + 4x + 1) \div (2x + 1).$$

(d) Use Anu's method to find

$$(x^4 + 3x^3 + 6x^2 + 5x + 3) \div (x^2 + x + 1).$$

Anu later tells use that she really was thinking of a  $1 \leftarrow 10$  system so that  $x$  does equal ten. Then her number  $2556_x$  really was two thousand, five hundred and fifty six and  $12_x$  really was twelve. Her statement:

$$(2x^3 + 5x^2 + 5x + 6) \div (x + 2) = 2x^2 + x + 3$$

is actually  $2556 \div 12 = 213$ .

(e) Check that  $2556 \div 12 = 213$  is correct in base 10.

(f) What division problems did you actually solve in parts (b), (c), and (d) in the  $1 \leftarrow 10$  system? Check that your answers are correct in this system.

**Hard Challenge!** Uh Oh! Anu has changed her mind. She now says she was thinking of a  $1 \leftarrow 11$  system.

Now  $2556_x$  means  $2 \cdot 11^3 + 5 \cdot 11^2 + 5 \cdot 11 + 6 = 3328_{\text{ten}}$ .

Similarly,  $12_x$  means  $1 \cdot 11 + 2 = 13_{\text{ten}}$ , and  $213_x$  means

$2 \cdot 11^2 + 1 \cdot 11 + 3 = 256_{\text{ten}}$ . So Anu's computation

$2556_x \div 12_x = 213_x$  is actually the (base 10) statement:

$$3328 \div 13 = 256.$$

(g) Check that  $3328 \div 13 = 256$  is also correct.

(h) What division problems did you actually solve for parts (b), (c), and (d) in the  $1 \leftarrow 11$  system? Check that they are correct.

**Problem 27.** This problem continues problem 26 above.  
Use Anu's method to show that

$$(x^4 + 4x^3 + 6x^2 + 4x + 1) \div (x + 1) = (x^3 + 3x^2 + 3x + 1).$$

- (a) What is this saying for  $x = 10$ ? Check that the division is correct.
- (b) What is this saying for  $x = 2$ ? Check that the division is correct.
- (c) What is this saying for  $x$  equal to each of 3, 4, 5, 6, 7, 8, 9, and 11? Check that each division is correct.
- (d) What is this saying for  $x = 0$ ?

PROBLEM BANK

**Problem 28.** Compute the following using dots and boxes:

$64212 \div 3$

$44793 \div 21$

$6182 \div 11$

$99916131 \div 31$

$637824 \div 302$

$2125122 \div 1011$

**Problem 29.**

(a) Fill in the squares using the digits 4, 5, 6, 7, 8, and 9 exactly one time each to make the largest possible sum:

$$\begin{array}{r} \square \quad \square \quad \square \\ + \square \quad \square \quad \square \\ \hline \end{array}$$

(b) Fill in the squares using the digits 4, 5, 6, 7, 8, and 9 exactly one time each to make the smallest possible (positive) difference:

$$\begin{array}{r} \square \quad \square \quad \square \\ - \square \quad \square \quad \square \\ \hline \end{array}$$

**Problem 30.** Make a base-6 addition table:

+	0	1	2	3	4	5
0						
1						
2						
3						
4						
5						

Use the table to solve the subtraction problems.

$13_{\text{six}} - 5_{\text{six}}$

$12_{\text{six}} - 3_{\text{six}}$

$10_{\text{six}} - 4_{\text{six}}$



**Problem 31.** Do these calculations in base 4. Don't translate to base 10 and then calculate there — try to work in base 4.

(1)  $33_{\text{four}} + 11_{\text{four}}$

(2)  $123_{\text{four}} + 22_{\text{four}}$

(3)  $223_{\text{four}} - 131_{\text{four}}$

(4)  $112_{\text{four}} - 33_{\text{four}}$

**Problem 32.** Make a base-5 multiplication table:

×	0	1	2	3	4
0					
1					
2					
3					
4					

Use the table to solve these division problems.

$11_{\text{five}} \div 2_{\text{five}}$

$22_{\text{five}} \div 3_{\text{five}}$

$13_{\text{five}} \div 4_{\text{five}}$

**Problem 33.**

(a) Here is a true fact in base five:

$$2_{\text{five}} \cdot 3_{\text{five}} = 11_{\text{five}}$$

Write the rest of this four fact family.

(b) Here is a true fact in base five:

$$13_{\text{five}} \div 2_{\text{five}} = 4_{\text{five}}$$

Write the rest of this four fact family.

**Problem 34** (AlphaMath). Letters stand for digits 0–9. In a given problem: the same letter always represents the same digit, and different letters always represent different digits. There is no relation between problems (so “A” in problem one and “A” in problem 3 might be different).

A

B

C

+ A

C

B

C

B

A

O

N

E

+ O

N

E

T

W

O

A

A

A

+

H

A

**Notes:** “O” represents the letter O and not the digit zero. Two and three digit numbers never start with the digit o.

**Problem 35.** Here’s another AlphaMath problem. (See problem 34 for the rules of AlphaMath.)

$$\begin{array}{r} \phantom{+} T \phantom{+} E \phantom{+} N \\ + \phantom{+} N \phantom{+} O \phantom{+} T \\ \hline N \phantom{+} I \phantom{+} N \phantom{+} E \end{array}$$

- (a) Solve this AlphaMath problem in base 10.
- (b) Now solve it in base 6.

**Problem 36.**Find all solutions to this AlphaMath problem in base 9. (See problem 34 for the rules of AlphaMath.) Note: even though this is two calculations, it is a *single problem*. All T’s in both calculations represent the same digit, all O’s represent the same digit, and so on.

$$\begin{array}{r} \phantom{-} T \phantom{-} O \\ - \phantom{-} B \phantom{-} E \\ \hline O \phantom{-} R \end{array} \qquad \begin{array}{r} \phantom{-} N \phantom{-} O \phantom{-} T \\ - \phantom{-} T \phantom{-} O \\ \hline B \phantom{-} E \end{array}$$

(Remember that “O” represents the letter O and not the digit zero, and that two and three digit numbers never start with the digit zero.)

**Problem 37.** A *perfect square* is a number that can be written as  $a \cdot a$  or  $a^2$  (some number times itself).

- (a) Which of the following *base seven numbers* are perfect squares? For each number, answer **yes** (it is a perfect square) or **no** (it is not a perfect square) and give a justification of your answer.

$$4_{\text{seven}} \qquad 25_{\text{seven}} \qquad 51_{\text{seven}}$$

- (b) For which choices of base  $b$  is the number  $100_b$  a perfect square? Justify your answer.

**Problem 38.** This is a single AlphaMath problem. (So all G’s represent the same digit. All A’s represent the same digit. And so on.)

Solve the problem in **base 6**.

$$GALON = (GOO)^2 \qquad ALONG = (OOG)^2$$

**Problem 39.** Some ink was spilled on these problems. The answers were correct. Can you determine the missing digits and the bases?



$$\begin{array}{r} \phantom{000}8 \\ + \phantom{000}1 \\ \hline 1000 \end{array}$$

$$\begin{array}{r} 100 \\ - \phantom{000}2 \\ \hline 1 \end{array}$$

**Problem 40.**

(a) Rewrite each subtraction problem as an addition problem:

$x - 156 = 279$

$279 - 156 = x$

$a - x = b$

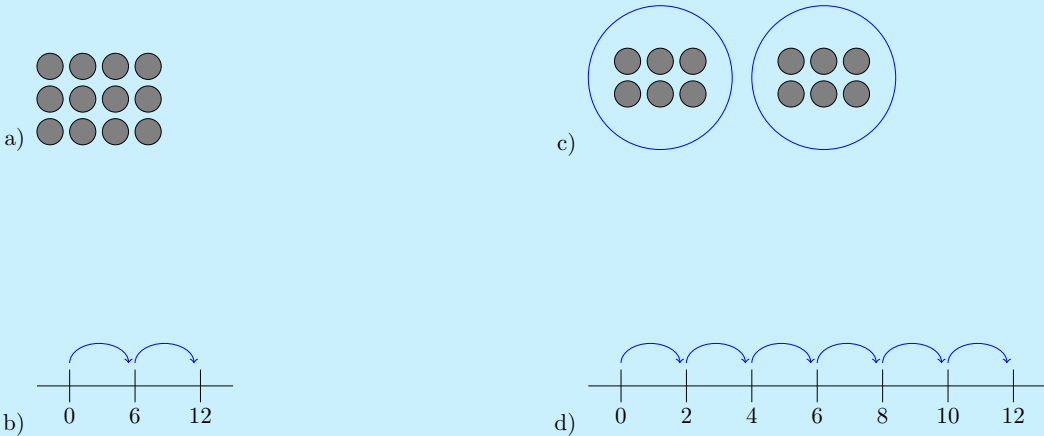
(b) Rewrite each division problem as a multiplication problem:

$24 \div x = 12$

$x \div 3 = 27$

$a \div b = x$

**Problem 41.** Which of the following models represent the same multiplication problem? Explain your answer.



**Problem 42.** Show an area model for each of these multiplication problems. Write down the standard computation next to the area model and see how it compares:

$20 \times 33$

$24 \times 13$

$17 \times 11$

**Problem 43.** Suppose the 2 key on your calculator is broken. How could you still use the calculator compute these products? Think about what properties of multiplication might be helpful. (Write out the calculation you would do on the calculator, not just the answer.)

$1592 \times 3344$

$2008 \times 999$

$655 \times 525$

**Problem 44.** Today is Jennifer's birthday, and she's twice as old as her brother. When will she be twice as old as him again? Choose the best answer and justify your choice.

- (1) Jennifer will always be twice as old as her brother.
- (2) It will happen every two years.
- (3) It depends on Jennifer's age.
- (4) It will happen when Jennifer is twice as old as she is now.
- (5) It will never happen again.

**Problem 45.**

(a) Find the quotient and remainder for each problem:

$$7 \div 3 \quad 3 \div 7 \quad 7 \div 1 \quad 1 \div 7 \quad 15 \div 5 \quad 8 \div 12$$

(b) How many possible remainders are there when dividing by these numbers? Justify what you say.

$$2 \quad 12 \quad 62 \quad 23$$

**Problem 46.** Identify each problem as either partitive or quotative division and say why you made that choice. Then solve the problem.

- (a) Adriana bought 12 gallons of paint. If each room requires three gallons of paint, how many rooms can she paint?
- (b) Chris baked 15 muffins for his family of five. How many muffins does each person get?
- (c) Prof. Davidson gave three straws for each student for an activity. She used 51 straws. How many students are in her class?

**Problem 47.** Use the digits 1 through 9. Use each digit exactly once. Fill in the squares to make all of the equations true.

$$\begin{array}{rcl} \square - \square & = & \square \\ \square \div \square & = & \square \\ \square + \square & = & \square \end{array}$$